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Conformal Mappings to Multiply Connected Polycircular Arc Domains

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Abstract. There has been much recent interest in finding analytical formulae for conformal mappings from canonical multiply connected circular regions to multiply connected polygonal regions. Such formulae are the multiply connected generalizations of the Schwarz-Christoffel formula of classical function theory. A natural generalization of polygonal domains is the class of polycircular arc domains whose boundaries are a union of circular arc segments. This paper describes a theoretical method for the construction of conformal mappings from multiply connected circular domains, of arbitrary finite connectivity, to conformally equivalent polycircular arc domains. This work generalizes results on the doubly connected case by Crowdy & Fokas [10].

Keywords. Conformal map, Schwarzian derivative, polycircular arc, Schottky-Klein prime function, automorphic function.

2000 MSC. 30C20, 30C30.

1. Introduction

A Schwarz-Christoffel mapping is a conformal mapping from a simple canonical domain, such as a unit disc, or the upper half plane, to a polygonal domain. Such mappings commonly arise in both theory and applications and they form the central topic of an excellent monograph by Driscoll & Trefethen [21]. The Schwarz-Christoffel mapping (henceforth, S-C mapping) to a simply connected polygonal domain dates back to the 1860's [21]; the formula for doubly connected domains was first derived by Akhiezer [2] (see also Komatu [29]). Such formulae are particularly useful because any simply connected shape can be well

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approximated by a polygon and the S-C mapping formula then provides a constructive approach to the conformal mapping problem. The natural question of generalizing this formula to polygonal domains of arbitrary finite connectivity remained open until recently. Progress in this direction was partly impeded by difficulties associated with solving the so-called "parameter problem", present even in the simply connected case, together with various other difficulties such as "crowding" [21]. These practical impediments have now been overcome and readily-transferable software operating on platforms such as `MATLAB` are available [22] in the simply connected case.

Prompted by these numerical developments, the long-standing theoretical problem of finding general formulae for S-C mappings to higher connected polygonal domains has recently been solved. DeLillo, Elcrat & Pfaltzgraff [19] were the first to produce a multiply connected S-C mapping formula from an unbounded circular region to the unbounded region exterior to a finite collection of polygonal objects. Their arguments rely on use of reflection principles and extend an approach to doubly connected S-C mappings given in [20]. The S-C formula to bounded multiply connected polygonal regions was first derived by Crowdy [7] who introduced novel aspects of classical function theory into the analysis of the problem. Indeed, employing the machinery of Schottky groups and the associated function theory, Crowdy was able to write a formula for the mapping, in a natural way, as the integral of a product of powers of a special transcendental function known as the *Schottky-Klein prime function* [3]. The approach extends naturally to unbounded polygonal regions [8]. A key theoretical ingredient in the derivation of the formulae in [7, 8] is the use of an intermediate conformal mapping from the original circular domain to a circular slit domain. Crowdy [9] has given a more geometrical derivation of the formulae in [7, 8] and, in particular, has emphasized the role of conformal slit mappings in the construction. DeLillo [18] has elucidated theoretical connections between the approaches in [20, 7, 8].

An extension of the theory of S-C mappings is the related theory of mappings to the so-called polycircular arc domains. These are domains whose boundaries are made up of arcs of circles. Straight line segments are a particular case of circular arcs so polycircular arc mappings include S-C mappings as a special case. Nehari [31] and Ablowitz & Fokas [1] discuss the general theory of conformal mappings to simply connected polycircular arc domains. A numerical construction of such mappings to simply connected regions has been carried out by Bjorstad & Grosse [6] & Howell [27]. In a recent paper, Crowdy & Fokas [10] have presented a constructive approach to finding the conformal maps from a concentric annulus in a complex preimage plane to any bounded doubly connected polycircular arc domain.

The present paper is a sequel to [10]; it generalizes the construction given there to the case of polycircular arc domains of any finite connectivity. For domains of connectivity greater than two, it is necessary to employ an intermediate conformal mapping to a circular slit domain to derive the relevant formulae — a device that also proved crucial in the construction of multiply connected S-C mappings in [7, 8]. This paper also follows that of [7, 8] in emphasizing the theoretical role played by the Schottky-Klein prime function [3, 26].

2. The conformal mapping problem

Let D_ζ be the unit disc in the ζ-plane with M smaller disjoint circular discs $D_1, \ldots, D_M$ excised. Let C_0 denote the unit circle and let $C_1, \ldots, C_M$ be the boundaries of the discs of $D_1, \ldots, D_M$. It is known from the multiply connected generalization of the Riemann Mapping Theorem [25] that any $(M+1)$-connected target domain D_z in a complex z-plane can be conformally mapped to some such circular region D_ζ for some choice of the parameters q_j, δ_j, $j = 1, \ldots, M$, known as the *conformal moduli*.

In this paper D_z is taken to be a bounded $(M+1)$-connected polycircular arc domain. It is a bounded $(M+1)$-connected region with $M+1$ boundaries each of which is a union of circular arc segments. Let P_j, $j = 0, 1, \ldots, M$, denote the boundaries of D_z corresponding to the images of the circles C_j, $j = 0, 1, \ldots, M$. For any choice of $j = 0, 1, \ldots, M$, let P_j be a union of n_j circular arc segments each defined by the equations

$$\left| z - \Delta_k^{(j)} \right|^2 = \left(Q_k^{(j)} \right)^2, \qquad k = 1, \ldots, n_j, \tag{1}$$

for some complex parameters $\Delta_k^{(j)}$, $k = 1, \ldots, n_j$, and some real parameters $Q_k^{(j)}$, $k = 1, \ldots, n_j$. Actually, our analysis carries over to cases in which certain portions of the boundary are straight-line segments but, for clarity of exposition, we focus on the case where all boundary segments satisfy (1). Define

$$N = \sum_{k=0}^{M} n_k.$$

The conformal mapping problem is to find the functional form of a conformal map from D_ζ to D_z. Let the points $z_k^{(j)}$, $j = 0, \ldots, M$, $k = 1, \ldots, n_j$, denote the vertices of D_z, i.e. the points at which the distinct circular arc segments making up the boundaries intersect. The prevertices on the circles C_j, $j = 0, 1, \ldots, M$, in the ζ-plane are defined to be the points $a_k^{(j)}$, $j = 0, \ldots, M$, $k = 1, \ldots, n_j$.

3. The circular slit domain

To proceed with the construction, it is expedient to introduce an intermediate η-plane. Consider a conformal mapping $\eta(\zeta)$ mapping D_ζ to a conformally equivalent circular slit domain D_η. Figure 1 shows a schematic in the triply connected

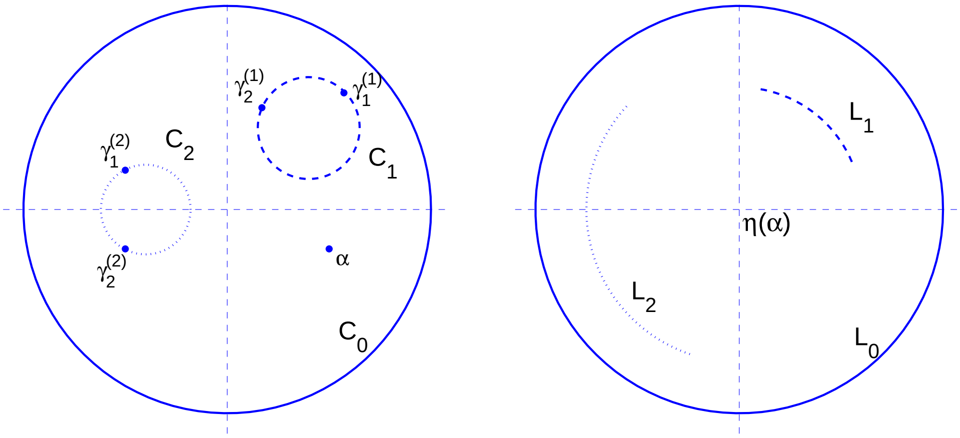

Figure 1. A typical circular slit mapping from a triply connected circular region D_ζ in a ζ-plane to a triply connected circular slit domain D_η in a η-plane. The point α in D_ζ maps to the origin in the η-plane. The points $\gamma_1^{(j)}, \gamma_2^{(j)}$, $j = 1, 2$, are also indicated.

case with $M = 2$. Let the image of C_0 under this mapping be the unit circle in the η-plane which will be called L_0. The M circles $C_1, \ldots, C_M$, have circular slit images, centred on $\eta = 0$, and labeled $L_1, \ldots, L_M$. Let the arc L_j be specified by the conditions

$$|\eta| = r_j, \quad \arg[\eta] \in [\phi_1^{(j)}, \phi_2^{(j)}], \qquad j = 1, \ldots, M.$$

There will be two preimage points on the circle C_j in the ζ-plane, for each $j = 1, \ldots, M$, corresponding to the two end-points of the circular slit L_j. These two preimage points, labeled $\gamma_1^{(j)}$ and $\gamma_2^{(j)}$, satisfy the conditions

$$\begin{aligned} &\eta\left(\gamma_1^{(j)}\right) = r_j e^{i\phi_1^{(j)}}, \quad \frac{d}{d\zeta}\eta\left(\gamma_1^{(j)}\right) = 0, \qquad j = 1, \ldots, M, \\ &\eta\left(\gamma_2^{(j)}\right) = r_j e^{i\phi_2^{(j)}}, \quad \frac{d}{d\zeta}\eta\left(\gamma_2^{(j)}\right) = 0, \qquad j = 1, \ldots, M. \end{aligned} \tag{2}$$

These $2M$ zeros of $d\eta/d\zeta$ are simple zeros because the argument changes by 2π at each of these points.

The conformal mapping to the domain D_z in the z-plane can be viewed either as a mapping from the preimage region D_ζ, or as a mapping from the preimage region D_η. We will use $Z(\eta)$ to denote the conformal mapping from D_η to D_z and then introduce the composite mapping

$$\mathcal{Z}(\zeta) \equiv Z(\eta(\zeta)),$$

which maps D_ζ to D_z. It is the function $\mathcal{Z}(\zeta)$ that we seek to find. Naturally, the prevertices $a_k^{(j)}$, $j = 0, \ldots, M$, $k = 1, \ldots, n_j$, introduced earlier satisfy the conditions

$$\mathcal{Z}\left(a_k^{(j)}\right) = z_k^{(j)}, \qquad k = 1, \ldots, n_j,\ j = 0, 1, \ldots, M.$$

4. Mathematical formulation

Suppose that the circular arc L_j in the η-plane maps to a general curve C (not necessarily a circular arc) in a z-plane as shown in Figure 2. Let $\hat{s}$ denote

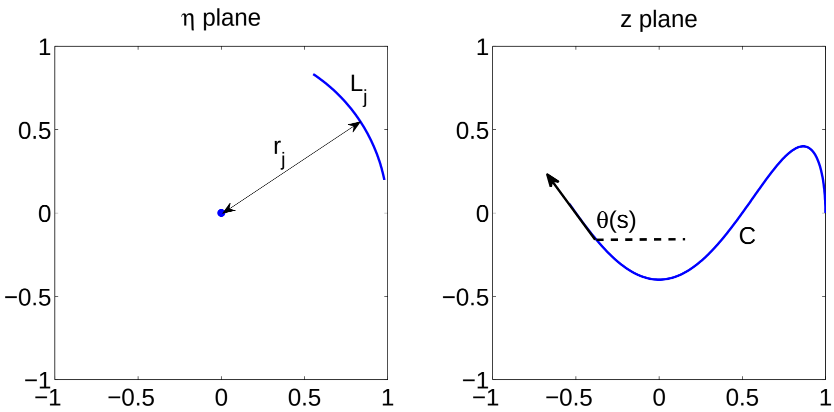

FIGURE 2. Under the conformal mapping $Z(\eta)$ the circular arc L_j, of radius r_j and centred at $\eta = 0$, is taken to map to a curve C in the complex z-plane. $\theta(s)$ is the angle made by the unit tangent at a point on C as a function of arclength s along it.

arclength along L_j and let s denote arclength along C. Since L_j is a circular arc of radius r_j centred at $\eta = 0$, it is clear that the complex unit tangent along L_j is

$$\frac{d\eta}{d\hat{s}} = \pm \frac{\mathrm{i}\,\eta}{r_j}, \tag{3}$$

where the choice of sign depends on the direction in which $\hat{s}$ is taken to increase around L_j. This choice of sign will not matter for our purposes. On the other hand, the complex unit tangent along C can be written as

$$\frac{dz}{ds} = e^{\mathrm{i}\,\theta(s)},$$

where $\theta(s)$ is defined as the angle made by the unit tangent with the positive real axis in the z-plane as shown in Figure 2.

It is convenient to consider the two quantities

$$\kappa = \frac{d\theta}{ds}(s), \qquad \frac{d\kappa}{ds} = \frac{d^2\theta}{ds^2}(s),$$

where κ is the curvature of C and $d\kappa/ds$ is its rate of change with respect to arclength. For cases in which C is either a circular arc or a straight line segment then, by definition, $d\kappa/ds = 0$ and this condition will provide the analytical criterion for the determination of the required conformal mapping. The advantage of introducing the intermediate η-plane will become clear in deriving this criterion.

Proposition 1. *The curvature κ of C is given, in terms of the conformal mapping function $Z(\eta)$, by*

$$\kappa = \pm\frac{1}{r_j|Z'(\eta)|}\left[1 + \mathrm{Re}\left(\frac{\eta Z''(\eta)}{Z'(\eta)}\right)\right], \tag{4}$$

where

$$Z'(\eta) = \frac{dZ}{d\eta}, \qquad Z''(\eta) = \frac{d^2Z}{d\eta^2}.$$

Proof. Since $dz/ds = e^{\mathrm{i}\,\theta(s)}$ it follows from the chain rule that

$$\kappa = \frac{d\theta}{ds} = \frac{\frac{d^2z}{ds^2}}{\mathrm{i}\,\frac{dz}{ds}}. \tag{5}$$

Since η lies on L_j, it also follows from the chain rule that

$$\frac{dz}{ds} = \frac{dZ}{d\eta}(\eta)\,\frac{d\eta}{ds} = \frac{dZ}{d\eta}(\eta)\,\frac{d\eta}{d\hat{s}}\,\frac{d\hat{s}}{ds} = Z'(\eta)\,\frac{d\eta}{d\hat{s}}\,\frac{d\hat{s}}{ds}. \tag{6}$$

But

$$ds = |dz| = |Z'(\eta)d\eta| = |Z'(\eta)||d\eta| = |Z'(\eta)|d\hat{s}, \tag{7}$$

which implies

$$\frac{d\hat{s}}{ds} = \frac{1}{|Z'(\eta)|}. \tag{8}$$

On use of (3) and (8) in (6), it is found that

$$\frac{dz}{ds} = \pm\frac{\mathrm{i}\,\eta Z'(\eta)}{r_j|Z'(\eta)|}.$$

Now we can write

$$\frac{d^2z}{ds^2} = \frac{\partial}{\partial\eta}\left(\pm\frac{\mathrm{i}\,\eta Z'(\eta)}{r_j|Z'(\eta)|}\right)\frac{d\eta}{ds} + \frac{\partial}{\partial\overline{\eta}}\left(\pm\frac{\mathrm{i}\,\eta Z'(\eta)}{r_j|Z'(\eta)|}\right)\frac{d\overline{\eta}}{ds},$$

and, hence,

$$\frac{\frac{d^2z}{ds^2}}{\frac{dz}{ds}} = \frac{\partial}{\partial\eta}\left(\pm\frac{\mathrm{i}\,\eta Z'(\eta)}{r_j|Z'(\eta)|}\right)\frac{\frac{d\eta}{ds}}{\frac{dz}{ds}} + \frac{\partial}{\partial\overline{\eta}}\left(\pm\frac{\mathrm{i}\,\eta Z'(\eta)}{r_j|Z'(\eta)|}\right)\frac{\frac{d\overline{\eta}}{ds}}{\frac{dz}{ds}}. \tag{9}$$

But, from (6),

$$\frac{\frac{d\eta}{ds}}{\frac{dz}{ds}} = \frac{1}{Z'(\eta)}, \qquad \frac{\frac{d\overline{\eta}}{ds}}{\frac{dz}{ds}} = -\frac{{r_j}^2}{\eta^2 Z'(\eta)}, \tag{10}$$

where, to derive the second equation, we have used the fact that $\overline{\eta} = {r_j}^2/\eta$ on L_j. A direct calculation gives the results

$$\begin{aligned} \frac{\partial}{\partial\eta}\left(\pm\frac{\mathrm{i}\,\eta Z'(\eta)}{r_j|Z'(\eta)|}\right) &= \pm\left(\frac{\mathrm{i}\,Z'(\eta)}{r_j|Z'(\eta)|} + \frac{\mathrm{i}\,\eta Z''(\eta)}{2r_j|Z'(\eta)|}\right), \\ \frac{\partial}{\partial\overline{\eta}}\left(\pm\frac{\mathrm{i}\,\eta Z'(\eta)}{r_j|Z'(\eta)|}\right) &= \pm\left(\frac{\mathrm{i}\,\eta Z'(\eta)\overline{Z''(\eta)}}{2r_j|Z'(\eta)|\overline{Z'(\eta)}}\right). \end{aligned} \tag{11}$$

Finally, substitution of (10) and (11) into (9) produces the result (4). ■

Proposition 2. *The quantity $d\kappa/ds$ is given, in terms of the conformal mapping function $Z(\eta)$, by*

$$\frac{d\kappa}{ds} = -\frac{\mathrm{Im}[\eta^2\{Z(\eta),\eta\}]}{{r_j}^2|Z'(\eta)|^2}, \tag{12}$$

where

$$\{Z(\eta),\eta\} \equiv \frac{Z'''(\eta)}{Z'(\eta)} - \frac{3}{2}\left(\frac{Z''(\eta)}{Z'(\eta)}\right)^2$$

denotes the Schwarzian derivative.

Proof. It is convenient to write

$$\frac{d\kappa}{ds} = \frac{\partial\kappa}{d\eta}\frac{d\eta}{ds} + \frac{\partial\kappa}{\partial\overline{\eta}}\frac{d\overline{\eta}}{ds}. \tag{13}$$

A direct calculation based on (4) leads to

$$\frac{\partial\kappa}{\partial\eta} = \pm\left(\frac{\eta}{2r_j|Z'(\eta)|}\{Z(\eta),\eta\} - \frac{\overline{\eta}|Z''(\eta)|^2}{4r_j|Z'(\eta)|^3}\right), \tag{14}$$

and, since κ is real, $\partial\kappa/\partial\overline{\eta}$ is given by the complex conjugate of this expression. It also follows, from (6), that

$$\frac{d\eta}{ds} = \pm\frac{\mathrm{i}\,\eta}{r_j|Z'(\eta)|}, \qquad \frac{d\overline{\eta}}{ds} = \mp\frac{\mathrm{i}\,\overline{\eta}}{r_j|Z'(\eta)|}. \tag{15}$$

Substitution of (14) and (15) into (13) gives the required result (12). ■

The result of Proposition 2 — that the rate of change of the curvature is connected to the Schwarzian derivative — is not new and a similar result is discussed in Beardon [5] (see also Needham [30]).

We now define the function

$$\mathcal{T}(\zeta) \equiv \eta^2\{Z(\eta),\eta\}, \tag{16}$$

where we now consider the function on the right hand side as a function of ζ. This function plays an important role in what follows. Since the image, under $Z(\eta)$, of every portion of each arc L_j for $j = 0, 1, \ldots, M$ is a circular arc, or a straight line segment, then $d\kappa/ds = 0$ on all the image curves. It follows, from Proposition 2, that for the particular case of conformal mappings to polycircular arc domains of interest, here we have

$$\overline{\eta^2\{Z(\eta), \eta\}} = \eta^2\{Z(\eta), \eta\} \qquad \text{on } L_j \text{ for } j = 0, 1, \ldots, M. \tag{17}$$

Equivalently,

$$\overline{\mathcal{T}(\zeta)} = \mathcal{T}(\zeta) \qquad \text{on } C_j \text{ for } j = 0, 1, \ldots, M. \tag{18}$$

The relations (18) will prove crucial in Section 7 for deriving the differential equation satisfied by the conformal mapping we seek.

The above derivation of conditions (17) has been geometrical in nature and relied on consideration of curvature and its relationship to the Schwarzian derivative. There is an alternative, purely algebraic method to derive the same result (17) and, for completeness, this is outlined in Appendix A.

To proceed further, we now introduce a special function called the *Schottky-Klein prime function* [3, 26]. By employing this function it is possible to obtain a convenient formula for $\eta(\zeta)$ while the functional form of $\mathcal{T}(\zeta)$ can also be derived up to a finite set of accessory parameters.

5. The Schottky-Klein prime function

This section gives a brief introduction to the Schottky-Klein prime function. We first define M Möbius maps ϕ_j, $j = 1, \ldots, M$, corresponding to the conjugation map for points on the circle C_j. That is, if C_j is determined by the equation

$$|\zeta - \delta_j|^2 = (\zeta - \delta_j)(\overline{\zeta} - \overline{\delta_j}) = {q_j}^2,$$

then

$$\overline{\zeta} = \overline{\delta_j} + \frac{{q_j}^2}{\zeta - \delta_j},$$

so that

$$\phi_j(\zeta) \equiv \overline{\delta_j} + \frac{{q_j}^2}{\zeta - \delta_j}. \tag{19}$$

If ζ is a point on C_j then its complex conjugate is given by $\overline{\zeta} = \phi_j(\zeta)$.

Next, introduce the Möbius maps

$$\theta_j(\zeta) \equiv \overline{\phi}_j(\zeta^{-1}) = \delta_j + \frac{{q_j}^2\zeta}{1 - \overline{\delta_j}\zeta}. \tag{20}$$

Let C_j' be the circle obtained by reflection of the circle C_j in the unit circle $|\zeta| = 1$ (i.e. the circle obtained by the transformation $\zeta \mapsto \overline{\zeta}^{-1}$).

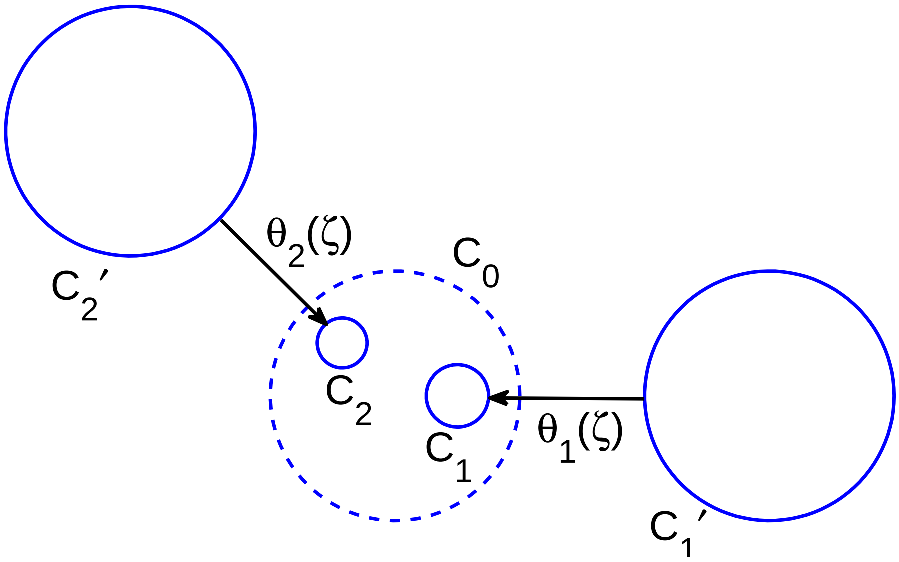

FIGURE 3. Schematic illustrating the circles C_j and C_j', and the Möbius maps $\theta_j(\zeta)$, in a triply connected case $M = 2$.

It can be easily verified that the image of the circle C_j' under the transformation θ_j is the circle C_j. Thus, θ_j identifies circle C_j' with circle C_j. Since the M circles $C_1, \ldots, C_M$ are non-overlapping, so are the M circles $C_1', \ldots, C_M'$. The classical *Schottky group* Θ is defined to be the infinite free group of mappings generated by compositions of the M basic Möbius maps θ_j, $j = 1, \ldots, M$, and their inverses θ_j^{-1}, $j = 1, \ldots, M$, and including the identity map.

Consider the (generally unbounded) region of the plane exterior to the $2M$ circles C_j, C_j', $j = 1, \ldots, M$. Let this region be called $\mathcal{F}$. $\mathcal{F}$ is known as the *fundamental region* associated with the Schottky group generated by the Möbius maps θ_j, $j = 1, \ldots, M$, and their inverses. This terminology is justified because the entire plane (excluding the limit points) is tessellated with copies of this fundamental region obtained by mapping $\mathcal{F}$ by the elements of the Schottky group. This fundamental region can be understood as having two "halves" — the half that is inside the unit circle but exterior to the circles C_j is the domain D_ζ, the other half is the region outside the unit circle and exterior to the circles C_j'. This other half (or copy of D_ζ) is obtained by an (antiholomorphic) reflection of D_ζ in the unit circle C_0. These two halves of $\mathcal{F}$, one just a reflection through the unit circle of the other, can be viewed as a model of the two "sides" of a compact (symmetric) Riemann surface associated with D_ζ known as its *Schottky double*.

Any compact Riemann surface of genus M also possesses exactly M holomorphic differentials [26] which will be denoted by $dv_j(\zeta)$, $j = 1, \ldots, M$. The functions $v_j(\zeta)$, $j = 1, \ldots, M$, are the *integrals of the first kind*; these functions are analytic, but not single-valued, everywhere in $\mathcal{F}$. We normalize the holomorphic differentials so that

$$\oint_{a_k} dv_j = \delta_{jk}, \qquad \oint_{b_k} dv_j = \tau_{jk},$$

where τ_{jk} is the so-called period matrix.

The following theorem is established in Hejhal [26]:

Theorem. *There is a unique function $X(\zeta,\gamma)$ defined by the properties:*

(i) $X(\zeta,\gamma)$ *is analytic everywhere in* $\mathcal{F}\times\mathcal{F}$.
(ii) *For* $\gamma\in\mathcal{F}$, $X(\zeta,\gamma)$ *has a second order zero at* γ *and at all of its equivalent points.*
(iii) *For* $\gamma\in\mathcal{F}$,

$$\lim_{\zeta\to\gamma}\frac{X(\zeta,\gamma)}{(\zeta-\gamma)^2}=1. \tag{21}$$

(iv) *For* $j=1,\dots,M$,

$$X(\theta_j(\zeta),\gamma)=\exp\big(-2\pi\,\mathrm{i}(2(v_j(\zeta)-v_j(\gamma))+\tau_{jj})\big)\frac{d\theta_j(\zeta)}{d\zeta}X(\zeta,\gamma). \tag{22}$$

The *Schottky-Klein prime function* is the square root of this function, i.e.

$$\omega(\zeta,\gamma)=(X(\zeta,\gamma))^{1/2},$$

where the branch of the square root is chosen so that $\omega(\zeta,\gamma)$ behaves like $(\zeta-\gamma)$ as $\zeta\to\gamma$. It is important to observe that $\omega(\zeta,\gamma)$ has a simple zero at $\zeta=\gamma$, and at all images of γ under the action of the group Θ.

Concerning the matter of evaluating the Schottky-Klein prime function, Baker [3] gives an infinite product formula for it, but precise convergence criteria for this product are not known. Even if it does converge, the evaluation of this infinite product becomes prohibitively slow as the connectivity of the domain increases. Crowdy & Marshall [15] have recently proposed a novel numerical scheme for the computation of the Schottky-Klein prime function which is not reliant on a sum or product over the Schottky group. This renders it much more efficient from a numerical standpoint. Moreover, freely downloadable software based on this algorithm have been made available [11].

6. A representation for $\eta(\zeta)$

It is necessary to find an explicit representation of the intermediate mapping $\eta(\zeta)$ from D_ζ to the circular slit domain D_η. This has been presented in [7, 14] and is given by

$$\eta(\zeta)=\frac{\omega(\zeta,\alpha)}{|\alpha|\omega(\zeta,\overline{\alpha}^{-1})}, \tag{23}$$

where $\omega(\cdot,\cdot)$ is the Schottky-Klein prime function associated with D_ζ. The parameter α is the point in D_ζ which maps to $\eta = 0$ and can be chosen arbitrarily. Once α has been chosen for a given D_ζ, then the parameters $\gamma_1^{(j)}, \gamma_2^{(j)}$, $j = 1, \ldots, M$, are determined via (23); that is, these parameters are purely functions of the parameters α and of q_j, δ_j, $j = 1, \ldots, M$.

7. Properties of $\mathcal{T}(\zeta)$

By the well-known chain rule for Schwarzian derivatives [30] it follows that

$$\left(\frac{d\eta}{d\zeta}\right)^2 \{Z(\eta), \eta\} = \{\mathcal{Z}(\zeta), \zeta\} - \{\eta(\zeta), \zeta)\}. \tag{24}$$

A proof of this result is a straightforward calculus exercise. In terms of $\mathcal{T}(\zeta)$, (24) can be written as

$$\{\mathcal{Z}(\zeta), \zeta\} = \frac{1}{\eta(\zeta)^2}\left(\frac{d\eta}{d\zeta}\right)^2 \mathcal{T}(\zeta) + \{\eta(\zeta), \zeta\}. \tag{25}$$

We now examine the singularities of $\mathcal{T}(\zeta)$. By making use of the fact that, near any prevertex $a_k^{(j)}$, the derivative of the conformal mapping must have the local behaviour

$$\frac{d\mathcal{Z}}{d\zeta} = \left(\zeta - a_k^{(j)}\right)^{\beta_k^{(j)}} h(\zeta)$$

for some parameter $\beta_k^{(j)}$, and a locally analytic function $h(\zeta)$ that is non-vanishing at $a_k^{(j)}$, it can be shown that, at each of the preimages $a_k^{(j)}$, $k = 1, \ldots, n_j$, $j = 0, 1, \ldots, M$, the quantity $\{\mathcal{Z}(\zeta), \zeta\}$ must have a second order pole. This same local argument is relevant even in the simply connected case [31, 1]. It then follows from (25) that $\mathcal{T}(\zeta)$ has second order poles at $a_k^{(j)}$, $k = 1, \ldots, n_j$, $j = 0, 1, \ldots, M$. Since $\{\mathcal{Z}(\zeta), \zeta\}$ is analytic at the points $\gamma_1^{(j)}, \gamma_2^{(j)}$, $j = 1, \ldots, M$, while $\{\eta(\zeta), \zeta\}$ has second order poles at these points, it also follows from (25) that $\mathcal{T}(\zeta)$ must have *fourth order* poles at $\gamma_1^{(j)}, \gamma_2^{(j)}$, $j = 1, \ldots, M$, because these are the simple zeros of $d\eta/d\zeta$. The strengths of these fourth order poles of $\mathcal{T}(\zeta)$ must be chosen so that there are *no* singularities of the right hand side of (25) at the points $\gamma_1^{(j)}, \gamma_2^{(j)}$, $j = 1, \ldots, M$.

We now return to the conditions (18) on $T(\zeta)$ derived earlier. In terms of the Möbius mappings just introduced, (18) implies that

$$\begin{aligned} \overline{\mathcal{T}}(\zeta^{-1}) &= \mathcal{T}(\zeta) && \text{on } C_0, \\ \overline{\mathcal{T}}(\phi_j(\zeta)) &= \mathcal{T}(\zeta) && \text{on } C_j \text{ for } j = 1, \ldots, M. \end{aligned} \tag{26}$$

But these functional relations can be analytically continued off the respective circles implying that

$$\overline{\mathcal{T}}(\zeta^{-1}) = \overline{\mathcal{T}}(\phi_j(\zeta)), \qquad j = 1, \ldots, M.$$

It then follows that

$$\mathcal{T}(\overline{\phi_j}(\zeta^{-1})) = \mathcal{T}(\theta_j(\zeta)) = \mathcal{T}(\zeta), \qquad \mathcal{T}(\theta_j^{-1}(\zeta)) = \mathcal{T}(\zeta), \tag{27}$$

or, since the Möbius maps $\theta_j(\zeta)$, $j = 1, \ldots, M$, and their inverses generate the Schottky group Θ, $\mathcal{T}(\zeta)$ is invariant under the action of the Schottky group.

The functional relations (27) can be used to argue that $\mathcal{T}(\zeta)$ is analytic in the fundamental region $\mathcal{F}$ associated with the Schottky group except for a finite set of poles on its boundary. A function that is invariant with respect to the action of a Schottky group, and is meromorphic in a fundamental region associated with the group, is known as an *automorphic function* [24, 4]. The right hand side of (25) therefore has a unique analytic continuation into D_ζ and (25) is the ordinary differential equation satisfied by $\mathcal{Z}(\zeta)$.

Combining all the information obtained so far, the function $\mathcal{T}(\zeta)$ must have the following properties:

(a) It must be an automorphic function with respect to the group Θ;
(b) it must satisfy the functional relation $\overline{\mathcal{T}}(\zeta^{-1}) = \mathcal{T}(\zeta)$;
(c) it must have second order poles at the prevertices $a_k^{(j)}$ and fourth order poles at the points $\gamma_1^{(j)}, \gamma_2^{(j)}$, $j = 1, \ldots, M$.

From the general theory of automorphic functions [24] it is known that there are a variety of ways to represent an automorphic function. In a quite separate application, Crowdy & Marshall [12, 16] have explored the effectiveness of several of these representations in the context of constructing the conformal mappings to so-called multiply connected quadrature domains. Such mappings are also automorphic functions under the action of a (finitely generated) Schottky group of precisely the kind considered here. The important point is that it is possible to write down a representation of the function $\mathcal{T}(\zeta)$ appearing in (25) up to a finite set of accessory parameters. These parameters must be found numerically as part of the construction of the mapping function.

Each choice of representation of $\mathcal{T}(\zeta)$ is likely to have advantages and disadvantages in terms of its numerical implementation (including the solution of the parameter problem), but such numerical matters will be left as a matter for future investigation. In Section 9, however, we will make a particular choice of a concrete representation of $\mathcal{T}(\zeta)$ in order to treat some explicit examples.

8. The parameter problem

It is important to check the consistency of the parameter problem. First, let us count the number of parameters needed to determine a given target domain. Any polycircular region with a total of N sides is characterized locally by $3N$ real parameters: the centre and radius of each circular arc must be specified. On the other hand, owing to a well-known invariance property of the Schwarzian derivative [31], the ordinary differential equation (25) for $\mathcal{Z}(\zeta)$ is invariant under transformations of the form

$$\mathcal{Z}(\zeta) \mapsto \frac{a\mathcal{Z}(\zeta)+b}{c\mathcal{Z}(\zeta)+d}, \qquad a,b,c,d \in \mathbb{C},\ ad-bc=1.$$

This invariance corresponds to 6 real degrees of freedom and these are associated with the 6 real constants that must be specified to uniquely solve the third order ordinary differential equation (25). It follows that (25) should depend on $3N-6$ specifiable parameters. We now verify this.

First, from the multiply connected generalization of the Riemann Mapping Theorem [25], the conformal mapping will depend on $3M-3$ real conformal moduli. These are the $3M$ parameters q_j, δ_j, $j=1,\dots,M$, minus 3 degrees of freedom associated with the Riemann Mapping Theorem. The choice of α in the intermediate mapping $\eta(\zeta)$ can be made arbitrarily, provided that it is strictly inside D_ζ, so it is excluded from the parameter count. Moreover, once α is chosen, then knowledge of the parameters q_j, δ_j, $j=1,\dots,M$, immediately determines the values of $\gamma_1^{(j)}, \gamma_2^{(j)}$, $j=1,\dots,M$. For this reason, the parameter α and the set $\gamma_1^{(j)}, \gamma_2^{(j)}$, $j=1,\dots,M$, are not included in the parameter count.

The function $\mathcal{T}(\zeta)$ has second order poles at the N prevertices to be determined. This represents only N real degrees of freedom because these poles are necessarily on the boundary circles of D_ζ so only their position on these circles is unknown. Counting multiplicities, and including the fourth order poles at $\gamma_1^{(j)}, \gamma_2^{(j)}$, $j=1,\dots,M$, $\mathcal{T}(\zeta)$ has a total of $2N+8M$ poles in the fundamental region $\mathcal{F}$ where we adopt the usual convention of only counting boundary singularities once. Since it is an automorphic function invariant under the action of a finitely generated Schottky group, by Abel's Theorem [3], $\mathcal{T}(\zeta)$ must also have $2N+8M$ (generally complex) zeros in the fundamental region. However, this represents only $2N+8M$ real degrees of freedom because the functional relation (b) means that if $\mathcal{T}(\zeta)$ has a zero at b strictly inside the fundamental region $\mathcal{F}$, then it necessarily has another zero at $1/\bar{b}$. Furthermore, given a function $\mathcal{T}(\zeta)$ with specified poles and zeros it can be multiplied by a real constant and will also satisfy all the requirements (a)–(c). This represents a further real degree of freedom. In summary, $\mathcal{T}(\zeta)$ depends on a total of

$$(3M-3)+N+(2N+8M)+1 = 11M+3N-2 \tag{28}$$

real parameters.

There are, however, constraints on these parameters. First, note from (25) that $\mathcal{T}(\zeta)$ must also have a second order zero at $\zeta=\alpha$ in order to remove a pole arising from the appearance of $\eta(\zeta)^2$ in the denominator of the multiplier of $\mathcal{T}(\zeta)$. This represents 4 real constraints. The second order poles of the right hand side of (25) at $\gamma_1^{(j)}, \gamma_2^{(j)}$, $j=1,\dots,M$, must be removable, leading to $4M$ real constraints. In the general case, the M complex constraints on the poles and zeros of an automorphic function (from Abel's Theorem [3]) give a total of $2M$ real constraints, but owing to the fact that, in the present case, all poles are on the boundary circles and that any zeros that are strictly inside the fundamental region come in pairs $(b, 1/\overline{b})$, these can be shown to amount to just M real constraints. Finally, a further $6M$ constraints follow from considering the fact that because the mapping $\mathcal{Z}(\zeta)$ is single-valued around the circle C_j, then on any closed curve inside D_ζ that encloses (and is close to) C_j we must have

$$\mathcal{Z}(\zeta) = \sum_{k=1}^{\infty} \frac{a_k}{(\zeta - \delta_j)^k} + \sum_{k=0}^{\infty} b_k (\zeta - \delta_j)^k,$$

for some set of coefficients a_k, b_k. This means that $\mathcal{Z}'''(\zeta)$ has a Laurent series representation of the form

$$\mathcal{Z}'''(\zeta) = -\sum_{k=1}^{\infty} \frac{k(k+1)(k+2)a_k}{(\zeta - \delta_j)^{k+3}} + \sum_{k=0}^{\infty} k(k-1)(k-2) b_k (\zeta - \delta_j)^{k-3}.$$

It follows that

$$\oint_{C_j} (\zeta - \delta_j)^k \mathcal{Z}'''(\zeta)\, d\zeta = 0, \qquad k = 0, 1, 2,\ j = 1, \ldots, M,$$

which, from (25), can be seen to constitute $6M$ real constraints on $\mathcal{T}(\zeta)$. In total we find

$$4 + 4M + M + 6M = 11M + 4 \tag{29}$$

real constraints. Subtracting (29) from (28) we conclude that equation (25) depends on

$$(11M + 3N - 2) - (11M + 4) = 3N - 6$$

free parameters.

In summary, it has been confirmed that the number of accessory parameters in (25) is as expected.

9. Examples

For a given target domain, it is generally necessary to solve numerically for the accessory parameters appearing in (25). To do so we need to write $\mathcal{T}(\zeta)$ explicitly by making some choice of its representation and then solving for the finite set of parameters appearing in this representation. As already mentioned, there are a variety of ways to represent automorphic functions with respect to a Schottky group (see Baker [3], Beardon [4], Crowdy & Marshall [12, 16]). Here, in order to simply corroborate the preceding theoretical formulation without delving into numerical issues arising from the solution of the parameter problem, we now apply it to some multiply connected polycircular arc mappings which happen to be known explicitly. The idea is that if $\mathcal{Z}(\zeta)$ is known then we can compute the quantity

$$\frac{\eta(\zeta)^2\left(\{\mathcal{Z}(\zeta),\zeta\}-\{\eta(\zeta),\zeta\}\right)}{\left(\frac{d\eta}{d\zeta}\right)^2} \tag{30}$$

directly. We can then demonstrate that this quantity is identically equal to an automorphic function $\mathcal{T}(\zeta)$ having all the requisite properties and for which we pose an alternative representation.

For present purposes, and following Baker [3] and Crowdy & Marshall [12], we elect here to represent $\mathcal{T}(\zeta)$ as a ratio of products of Schottky-Klein prime functions. All evaluations of the prime function in this section have been performed using the numerical codes prepared by Crowdy & Green [11].

9.1. An example of a doubly connected domain. Crowdy & Fokas [10] did not make use of an intermediate circular slit map in their derivation of polycircular arc mappings for doubly connected domains (it is not necessary in that case). It is therefore reassuring to check that the foregoing theory, which *does* use the intermediate slit mapping, applies to this case. As shown in an appendix in [10] the conformal mapping from $\rho < |\zeta| < 1$ to the domain D_z comprising the unit disc with a symmetric slit along the real axis is

$$\mathcal{Z}(\zeta)=\frac{P(-\zeta,\rho)-P(\zeta,\rho)}{P(-\zeta,\rho)+P(\zeta,\rho)},$$

where

$$P(\zeta,\rho)\equiv(1-\zeta)\prod_{j=1}^{\infty}\left(1-\rho^{2j}\zeta\right)\left(1-\rho^{2j}\zeta^{-1}\right).$$

The function $P(\zeta,\rho)$ is proportional to the Schottky-Klein prime function for the annulus. The circular slit mapping is given in this case by

$$\eta(\zeta)=\frac{|\alpha|P(\zeta\alpha^{-1},\rho)}{P(\zeta\overline{\alpha},\rho)}.$$

Taking a real value of α implies that the circular slit will be up-down symmetric about the real axis. The preimage domain D_ζ and the target domain D_z are also up-down symmetric. This means that the distribution of zeros of $\mathcal{T}(\zeta)$ is expected to share this up-down symmetry.

It has been verified that a representation for $\mathcal{T}(\zeta)$ is

$$\mathcal{T}(\zeta)=\frac{RP^2(\zeta\alpha^{-1},\rho)P^2(\zeta\overline{\alpha},\rho)\prod_{j=1}^4 P(\zeta a_j^{-1},\rho)\prod_{j=1}^4 P(\zeta\overline{a_j}^{-1},\rho)}{P^2(\zeta\rho^{-1},\rho)P^2(-\zeta\rho^{-1},\rho)P^2(\zeta\gamma^{-1},\rho)P^2(\zeta\overline{\gamma}^{-1},\rho)P^2(\zeta\gamma,\rho)P^2(\zeta\overline{\gamma},\rho)}, \tag{31}$$

where R is a constant, $\{a_j\}_{j=1}^4$ are simple zeros and γ is the preimage in D_ζ of the end of the circular slit in the upper half-plane of D_η (the preimage of the end of the slit in the lower half-plane is at $\overline{\gamma}$). For a given value of ρ the value of γ

can be found numerically by using Newton's method to find the zero of $d\eta/d\zeta$ on $|\zeta| = \rho$. The zeros of (30) were also found numerically. Then the zeros of the function in (31) can be found. After determining these zeros, an arbitrary value of ζ is picked and equation (31) evaluated at this point to provide a formula for the only remaining unknown parameter R. It was then verified that functional relation (31) holds identically for any other choice of ζ in the fundamental region. This provides a check of our general theoretical formulation.

Note that if the conformal mapping had not been known *a priori* the strategy would have been to pose an ansatz for $\mathcal{T}(\zeta)$ of the functional form (31) and use geometrical information on the target domain, as well as the other constraints on parameters discussed in Section 8, to solve for the parameters in (31).

Another check on the formulation comes from noting that the conformal mapping $\mathcal{Z}(\zeta)\colon D_\zeta \to D_z$ should not depend on the choice of α used in the circular slit mapping to D_η. To verify this, define

$$S(\zeta,\alpha) \equiv \{\eta(\zeta),\zeta\} + \frac{\eta(\zeta)^2\mathcal{T}(\zeta)}{\left(\frac{d\eta}{d\zeta}\right)^2}.$$

Since $S(\zeta,\alpha) = \{\mathcal{Z}(\zeta),\zeta\}$ then given any two different choices of α, say α_1 and α_2, then it should be expected that

$$\frac{S(\zeta,\alpha_1)}{S(\zeta,\alpha_2)} \equiv 1.$$

This identity was confirmed numerically (by evaluating at arbitrarily chosen values of ζ) for the choices $\alpha_1 = -\sqrt{0.2}$ and $\alpha_2 = 0.5$. The parameters in the case $\rho = 0.2$ with $\alpha = -\sqrt{0.2}$ (left table) and $\alpha = 0.5$ (right table) are shown in Table 1 (correct to 6 decimal places).

a_1	$+0.688977 + 0.724783\,\mathrm{i}$
a_2	$-0.758396 + 0.651794\,\mathrm{i}$
a_3	$+0.061944 + 0.190165\,\mathrm{i}$
a_4	$-0.178546 + 0.090119\,\mathrm{i}$
R	$-71.207654 + 0.000000\,\mathrm{i}$

a_1	$+0.754713 + 0.656055\,\mathrm{i}$
a_2	$-0.688141 + 0.725577\,\mathrm{i}$
a_3	$+0.176622 + 0.093834\,\mathrm{i}$
a_4	$-0.070189 + 0.187280\,\mathrm{i}$
R	$-102.917735 + 0.000000\,\mathrm{i}$

TABLE 1.

Finally, local arguments imply that near one of the prevertices, say $\zeta = \rho$,

$$\frac{d\mathcal{Z}}{d\zeta} = (\zeta-\rho)^\beta h(\zeta),$$

where $h(\zeta)$ is analytic and non-vanishing at ρ. The turning angle corresponding to prevertex $\zeta = \rho$ is π implying that $\beta = 1$. From this, it is a simple exercise to show that, near $\zeta = \rho$, the Schwarzian derivative must have the local behaviour

$$\{\mathcal{Z}(\zeta), \zeta\} = -\frac{3}{2}\frac{1}{(\zeta - \rho)^2} + \text{analytic function.}$$

As a final check on the formulation, we used the alternative representation of $\{\mathcal{Z}(\zeta), \zeta\}$ as given by the right hand side of (25) together with the expression (31) for $\mathcal{T}(\zeta)$ and confirmed that the second pole at $\zeta = \rho$ indeed has strength $-3/2$.

9.2. An example of a triply connected domain. Consider the lens-type region D_z shown in Figure 4 bounded by two circular arcs of the same radius, intersecting at ± 1, and two symmetrical slits along the real axis. The conformal mapping between the triply connected circular domain D_ζ and D_z is given by the following sequence of conformal mappings:

$$g(\zeta) = -\frac{\omega(\zeta, -1)}{\omega(\zeta, 1)}, \qquad \phi(g) = g^\beta, \qquad \eta(\phi) = \frac{\phi - 1}{\phi + 1}. \tag{32}$$

This sequence of mappings is illustrated in Figure 4. The parameter β is related to the angle at the two corners of the lens-type region. The composed mapping $\mathcal{Z}\colon D_\zeta \to D_z$ is

$$\mathcal{Z}(\zeta) = \eta(\phi(g(\zeta))).$$

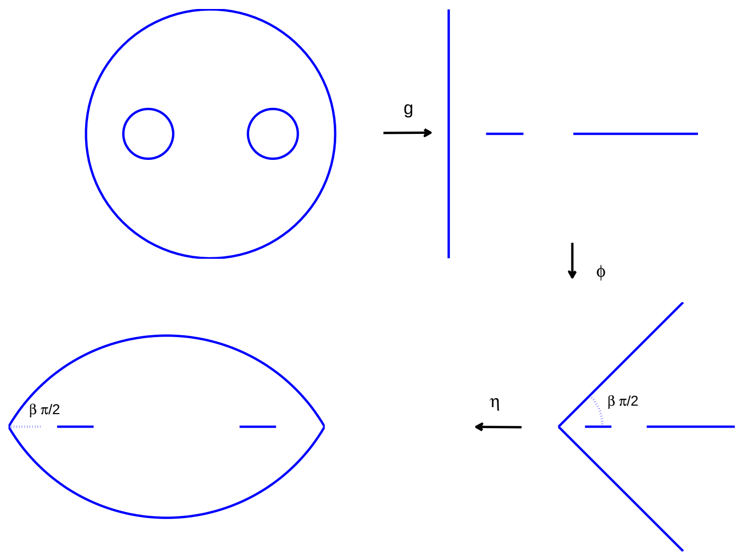

FIGURE 4. Sequence of conformal mappings (32) for the construction of an analytical expression for a triply connected lens domain.

By considerations similar to those in Section 9.1 we have verified that one representation for $\mathcal{T}(\zeta)$ is

$$\mathcal{T}(\zeta) = R\frac{T_n(\zeta)}{T_d(\zeta)}$$

where R is a real constant, while $T_n(\zeta)$ and $T_d(\zeta)$ are defined as follows:

$$T_n(\zeta) = \omega^2(\zeta,\alpha)\omega^2(\zeta,\overline{\alpha}^{-1})\prod_{j=1}^{2}\omega(\zeta,a_j)\omega(\zeta,a_j^{-1}) \times\prod_{j=3}^{5}\omega(\zeta,a_j)\omega(\zeta,a_j^{-1})\omega(\zeta,\overline{a_j})\omega(\zeta,\overline{a_j}^{-1})\prod_{j=6}^{9}\omega(\zeta,a_j)\omega(\zeta,\overline{a_j}),$$

and

$$T_d(\zeta) = \prod_{j=1}^{2}\omega^2(\zeta,\gamma_j)\omega^2(\zeta,\overline{\gamma_j}^{-1})\prod_{j=1}^{2}\omega^2(\zeta,\overline{\gamma_j})\omega^2(\zeta,\gamma_j^{-1})\prod_{j=1}^{6}\omega^2(\zeta,b_j).$$

γ_1,γ_2 are the preimages in D_ζ of the upper ends of both the circular slits in D_η. The parameters $a_1,\ldots,a_9$ are simple zeros which were determined numerically by solving for the zeros of (30). For the choices of β considered here it is found that two of the zeros, labeled a_1 and a_2, are purely real, three other zeros, labeled a_3,a_4,a_5, lie strictly inside the fundamental domain, while the zeros a_6,a_7,a_8,a_9 lie on either C_1 or C_2. We also have $b_1=\delta+q=-b_2$, $b_3=-\delta+q=-b_4$, and $b_5=1=-b_6$.

For δ=0.5, $q=0.1$, and $\alpha=0.2$ we find the following tables of parameter values for $\beta=0.25$ (left table) and $\beta=0.5$ (right table):

a_1	$+0.802802+0.000000\,\mathrm{i}$
a_2	$-0.806078+0.000000\,\mathrm{i}$
a_3	$-0.001643+0.145736\,\mathrm{i}$
a_4	$-0.664638+0.479002\,\mathrm{i}$
a_5	$+0.647050+0.470597\,\mathrm{i}$
a_6	$-0.422109+0.062713\,\mathrm{i}$
a_7	$-0.562922+0.077723\,\mathrm{i}$
a_8	$+0.552597+0.085051\,\mathrm{i}$
a_9	$+0.416578+0.055143\,\mathrm{i}$
R	$-0.115884+0.000000\,\mathrm{i}$

a_1	$+0.812604+0.000000\,\mathrm{i}$
a_2	$-0.815933+0.000000\,\mathrm{i}$
a_3	$-0.001570+0.157708\,\mathrm{i}$
a_4	$-0.718085+0.535868\,\mathrm{i}$
a_5	$+0.691757+0.521101\,\mathrm{i}$
a_6	$-0.422136+0.062747\,\mathrm{i}$
a_7	$-0.562716+0.077889\,\mathrm{i}$
a_8	$+0.552265+0.085255\,\mathrm{i}$
a_9	$+0.416590+0.055162\,\mathrm{i}$
R	$-0.129490+0.000000\,\mathrm{i}$

TABLE 2.

Numerical values are reported correct to 6 decimal places. Furthermore, for $\beta=0.5$, the parameters were determined for the two choices $\alpha=\alpha_1=0.2$ and

$\alpha = \alpha_2 = -0.75$ and, by evaluating at arbitrarily chosen values of ζ, it was verified numerically that

$$\frac{S(\zeta,\alpha_1)}{S(\zeta,\alpha_2)} \equiv 1,$$

indicating that $\mathcal{T}(\zeta)$ is indeed independent of the choice of α.

10. Discussion

The principal result of this paper is to show that the conformal map $\mathcal{Z}(\zeta)$ from a circular preimage region D_ζ to a given multiply connected polycircular arc domain is a solution of the third order ordinary differential equation

$$\{\mathcal{Z}(\zeta),\zeta\} = \frac{1}{\eta(\zeta)^2}\left(\frac{d\eta}{d\zeta}\right)^2 \mathcal{T}(\zeta) + \{\eta(\zeta),\zeta\}, \tag{33}$$

where curly brackets denote the Schwarzian derivative. Furthermore it has been shown that the right-hand side of (33) can be written explicitly up to a finite set of accessory parameters that must be determined as part of the solution (the "parameter problem"). An explicit form for $\eta(\zeta)$ is given by (23) in terms of the Schottky-Klein prime function, while a functional form for $\mathcal{T}(\zeta)$ can be determined by using standard techniques for representing automorphic functions invariant under the action of a classical, finitely generated Schottky group. The numerical challenges associated with the solution of the parameter problem are a topic for future investigation.

It is interesting to point out that, just as in the simply connected case, the linearization of the differential equation (33) yields a second order ordinary differential equation [1] and $\mathcal{Z}(\zeta)$ can be expressed as a ratio of two solutions of this equation. The resulting second order linear ordinary differential equations can be considered as generalizations, to higher genus Riemann surfaces, of well-known equations arising in the theory of conformal mapping to simply connected polycircular domains such as the *hypergeometric equation* [31] when the domain has just three singular points, or the *Heun equation* [23] when there are four singular points.

Appendix A. Alternative derivation of (17)

In this appendix we outline a purely algebraic method to derive the result (17). On the k-th circular arc of the j-th boundary, it follows from (1) that

$$\overline{z} - \overline{\Delta_k^{(j)}} = \frac{[Q_k^{(j)}]^2}{z - \Delta_k^{(j)}}, \qquad j = 0, 1, \ldots, M. \tag{34}$$

This formula defines $\overline{z}$ as a function of z on this curve; such a function is usually dubbed the *Schwarz function* of the curve [17, 30]. Define

$$S_k^{(j)}(z) = \overline{\Delta_k^{(j)}} + \frac{[Q_k^{(j)}]^2}{z - \Delta_k^{(j)}}, \qquad j = 0, 1, \ldots, M, \tag{35}$$

to be the Schwarz function of the k-th circular arc of the j-th boundary. This function clearly has a meromorphic continuation off the curve and there is no ambiguity in taking derivatives, with respect to z, of these local Schwarz functions. On differentiation of (35) with respect to z, and use of (35), we can write

$$\frac{dS_k^{(j)}(z)}{dz} = -\frac{S_k^{(j)}(z) - \overline{\Delta_k^{(j)}}}{z - \Delta_k^{(j)}}. \tag{36}$$

Differentiating (36) with respect to z yields

$$\frac{d^2 S_k^{(j)}(z)}{dz^2} = -\frac{\frac{dS_k^{(j)}}{dz}(z)}{z - \Delta_k^{(j)}} + \frac{S_k^{(j)}(z) - \overline{\Delta_k^{(j)}}}{(z - \Delta_k^{(j)})^2} = 2\frac{S_k^{(j)}(z) - \overline{\Delta_k^{(j)}}}{(z - \Delta_k^{(j)})^2}. \tag{37}$$

A ratio of equations (36) and (37) implies that

$$\frac{\frac{dS_k^{(j)}}{dz}(z)}{\frac{d^2 S_k^{(j)}}{dz^2}(z)} = -\frac{1}{2}(z - \Delta_k^{(j)}), \tag{38}$$

so that a further differentiation of (38) with respect to z leads to

$$-\frac{d^3 S_k^{(j)}(z)}{dz^3} + \frac{3}{2}\frac{\left(\frac{d^2 S_k^{(j)}}{dz^2}(z)\right)^2}{\frac{dS_k^{(j)}}{dz}(z)} = 0. \tag{39}$$

It is important to notice that this third order ordinary differential equation for the Schwarz function of the k-th circular arc of the j-th boundary does not depend explicitly on either index j or k; thus, the local Schwarz function of *every* circular arc segment of the boundary of the polycircular arc domain satisfies the *same* differential equation (39). Also note that if a portion of the boundary is a straight line segment, so that its Schwarz function is linear, then it also satisfies (39).

While the analytic relation (39) is valid off the original curve it can be evaluated *on* the k-th circular arc of the j-th boundary and then we have, in terms of the conformal mapping $Z(\eta)$ from the circular slit domain D_η, the relations

$$z = Z(\eta), \qquad S_k^{(j)}(z) = \overline{Z(\eta)} = \overline{Z}\left(\frac{{r_j}^2}{\eta}\right), \tag{40}$$

where η lies on L_j. On substitution of the expressions (40) into (39), after some algebra, it can be shown that (39) reproduces condition (17).

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