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Green's function for the Laplace–Beltrami operator on the surface of a three-dimensional ring torus is constructed. An integral ingredient of our approach is the stereographic projection of the torus surface onto a planar annulus. Our representation for Green's function is written in terms of the Schottky–Klein prime function associated with the annulus and the dilogarithm function. We also consider an application of our results to vortex dynamics on the surface of a torus.

1. Introduction

The subject of potential theory on surfaces is of interest across many areas in the mathematical sciences both from a purely abstract perspective and in the context of numerous diverse physical applications. Of fundamental importance to this subject is Green's function for the Laplace–Beltrami operator on the surface. For both analytical and computational purposes, it is valuable to have explicit representations for this function.

The simplest case is that of the Euclidean plane, and for this, Green's function is of course elementary. Green's function is also well-known for the case of the sphere; explicit representations are presented, for example, in Kimura & Okamoto [1], Kimura [2], and also Crowdy & Cloke [3]. Kimura [2] also considers the case of the hyperbolic plane. However, for more complex surfaces, for example of higher genus or non-constant curvature, the theory is mathematically more complicated and far fewer explicit results are known. Perhaps the simplest such surface is that of the ring torus, and this is therefore a natural first choice on which to attempt to extend the existing theory.

In this paper, we construct an explicit representation for Green's function for the Laplace–Beltrami operator on a toroidal surface. This representation is as a function of

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a single complex variable. To the best of the authors' knowledge, this is the first such explicit formula to be constructed for this function.

We have chosen deliberately to address this issue from the most general potential theoretical viewpoint, with the intention that the mathematics developed herein can be readily applied elsewhere to solve problems on the torus for which the Laplace–Beltrami equation turns out to be the governing equation. Understanding potential theory on surfaces such as the torus could yield valuable insight into interesting phenomena occurring in numerous areas, including, for example, vortex dynamics. Indeed, it is primarily within this context that [1–3] all arose. Much of the existing theory regarding vortex dynamics on curved surfaces pertains to compact surfaces of genus zero and in particular the sphere, owing largely to the desire to understand various physical phenomena that occur on the Earth's surface; indeed, a vast body of work has emerged since the late 1970s involving various models of systems of point vortices on the sphere beginning with Bogomolov [4], and including Crowdy & Cloke [3], Kimura & Okamoto [1], Boatto & Simó [5] and Kidambi & Newton [6,7]. However, fewer results are known explicitly for more general surfaces. In addition to Kimura [2], Kim [8] has analysed the case of the spheroid, while both Hally [9] and Boatto & Koiller [10] have considered point vortex motion on general curved surfaces from a more abstract point of view. As stated in [2], results concerning more general curved surfaces are likely to find relevance beyond the realms of classical fluid mechanics. We mention, for example, quantum mechanics and flows of superfluid films [11]. Indeed, Corrada–Emmanuel [12] has found an exact solution for the velocity field for so-called superfluid film vortices on the surface of a torus. Another potential application of the results in this paper is in the so-called 'best-packing' problem for points on a compact surface; this problem arises in numerous areas, including statistical sampling and computer-aided design. Hardin & Saff [13] have analysed this problem on the torus, in addition to other compact surfaces. Recently, Newton & Sakajo [14] have investigated the link between these optimum point distributions and point vortex equilibria for the case of the sphere. It may be possible to conduct an analysis similar to that of Newton & Sakajo [14] on a toroidal surface.

Our method can be summarized as follows. At the heart of the approach used by Crowdy & Cloke [3] is the stereographic projection of the sphere onto the complex plane. Other authors including [2,7,8] also use stereographic projection from the surface in consideration to the complex plane. Being cast in terms of a single complex variable turns out to be a particularly expedient approach, in addition to being mathematically elegant. Indeed, viewing the surface of the sphere in this way, Crowdy and co-workers have been able to discover classes of analytical solutions describing various vortical structures on the sphere (for example, in addition to [3], see [15–17]). In the light of these advantages, we have chosen to couch our work in a similar complex variable framework. We introduce a stereographic projection of the surface of the torus onto a concentric annulus in the plane, and show how to write the Laplace–Beltrami operator on the torus in terms of the complex coordinates in this plane, and then integrate a corresponding Laplace–Beltrami equation to obtain Green's function associated with the torus. We remark that Corrada-Emmanuel [12] also employs the technique of stereographic projection to map the torus surface onto a planar rectangular cell.

In terms of our complex variable, Green's function must be doubly periodic. Similar properties are exhibited by Green's functions in doubly periodic lattices of parallelograms or rectangles in the plane [18,19] and also arise in studying point vortex dynamics in such domains [20–23]; these rectangular cells are in fact often dubbed 'flat tori'. Solutions for these are commonly constructed in terms of elliptic functions in order to capture their required double periodicity. More recently, Crowdy [24] has dealt with such a problem by transforming the rectangular cells to concentric annuli and by using a special function, related to elliptic functions, known as the Schottky–Klein prime function. We will use this function to construct our doubly periodic solution for Green's function associated with the torus. We also appeal to another special function known as the dilogarithm function [25]. It is important to bear in mind that although we address the present problem in terms of a planar variable, the intrinsic curvature effects of the three-dimensional toroidal surface are fully incorporated into our theoretical framework.

To complete the paper, we choose to illustrate an application of our theory from a perspective in vortex dynamics, motivated by extending the existing work in this area on the sphere referred to above. In this context, Green's function is equivalent to the streamfunction for the flow. The representation we derive for this function can be viewed as the analogue on the torus of that presented by Crowdy & Cloke [3] on the sphere.

2. Formulation of problem

Let $\mathcal{T}_{R,r}$ denote the surface of the ring torus consisting of points $(x, y, z) \in \mathbb{R}^3$ where

$$x = (R - r\cos\theta)\cos\phi, \quad y = (R - r\cos\theta)\sin\phi \quad \text{and} \quad z = r\sin\theta. \tag{2.1}$$

Here, R and r are the major and minor radii, respectively, $R > r$, and $\theta, \phi \in [0, 2\pi]$. $\mathcal{T}_{R,r}$ is formed by taking a circle C of radius r centred a distance R from the origin in the (x, z)-plane, and rotating it through 2π about the z-axis. θ denotes the angle around the circle C, and ϕ denotes the azimuthal angle about the z-axis. It can be shown that the Laplace–Beltrami operator on the surface of the torus $\mathcal{T}_{R,r}$ in terms of the coordinates (2.1) is given by

$$\nabla^2_{\mathcal{T}_{R,r}} \equiv \frac{1}{r^2(R - r\cos\theta)}\frac{\partial}{\partial\theta}\left((R - r\cos\theta)\frac{\partial}{\partial\theta}\right) + \frac{1}{(R - r\cos\theta)^2}\frac{\partial^2}{\partial\phi^2}. \tag{2.2}$$

We seek Green's function $\psi(\theta, \phi)$ for the operator $\nabla^2_{\mathcal{T}_{R,r}}$. This is a real-valued function and satisfies the equation

$$\nabla^2_{\mathcal{T}_{R,r}}\psi(\theta, \phi) = \delta(\theta, \phi; \theta_0, \phi_0) - \frac{1}{4\pi^2 rR} \tag{2.3}$$

where $\delta(\theta, \phi; \theta_0, \phi_0)$ represents the Dirac delta function. We remark that the additive constant term on the right-hand side of (2.3) is required by Gauss's divergence theorem, and that $4\pi^2 rR$ is the surface area of the torus $\mathcal{T}_{R,r}$. Note that we could have chosen any function to add to the right-hand side of (2.3) provided Gauss's divergence theorem is satisfied, but our choice of the constant $-1/4\pi^2 rR$ is the simplest and turns out to be meaningful in numerous physical problems where this Green's function arises.

We shall solve for $\psi(\theta, \phi)$ by transforming the problem to a complex plane, as follows.

3. Stereographic projection

The stereographic projection from the torus $\mathcal{T}_{R,r}$ to a rectangle in a complex Z-plane is presented in Akhiezer [26]:

$$Z = \phi + \mathrm{i}\int_0^\theta \frac{\mathrm{d}\theta'}{\alpha - \cos\theta'}, \quad \text{where } \alpha = \frac{R}{r}. \tag{3.1}$$

This map is one–one and conformal, and the dimensions of the rectangle are 2π and L, where

$$L = \int_0^{2\pi} \frac{\mathrm{d}\theta'}{\alpha - \cos\theta'} = 2\pi\mathcal{A}, \quad \text{where } \mathcal{A} = \frac{1}{\sqrt{\alpha^2 - 1}}. \tag{3.2}$$

It is well known that the conformal map from this rectangle to a concentric annulus $D_\zeta = \{\zeta \,|\, \rho \le |\zeta| \le 1\}$, where $\rho = \mathrm{e}^{-L}$ is

$$\zeta(Z) = \mathrm{e}^{\mathrm{i}Z}. \tag{3.3}$$

Thus, the stereographic projection of $\mathcal{T}_{R,r}$ to D_ζ can be written in the form

$$\zeta = q(\theta)\,\mathrm{e}^{\mathrm{i}\phi}, \tag{3.4}$$

where

$$q(\theta) = \exp\left(-\int_0^\theta \frac{\mathrm{d}\theta'}{\alpha - \cos\theta'}\right) \tag{3.5}$$

and

$$\int_0^\theta \frac{\mathrm{d}\theta'}{\alpha - \cos\theta'} = 2\mathcal{A}\tan^{-1}\left(\sqrt{\frac{\alpha+1}{\alpha-1}}\tan\left(\frac{\theta}{2}\right)\right). \tag{3.6}$$

Equation (3.4) is analogous in functional form to the stereographic projection to the plane of the sphere considered by Crowdy & Cloke [3]. Note that $\zeta \mapsto \mathrm{e}^{2\pi \mathrm{i}}\zeta$ corresponds to a rotation through 2π in the ϕ-direction on $\mathcal{T}_{R,r}$, while $\zeta \mapsto \rho\zeta$ corresponds to a rotation through 2π in the θ-direction on $\mathcal{T}_{R,r}$. Using the facts that

$$\left.\frac{\partial}{\partial\theta}\right|_\phi \equiv -\frac{1}{\alpha-\cos\theta}\left(\zeta\left.\frac{\partial}{\partial\zeta}\right|_{\bar\zeta} + \bar\zeta\left.\frac{\partial}{\partial\bar\zeta}\right|_\zeta\right) \quad \text{and} \quad \left.\frac{\partial}{\partial\phi}\right|_\theta \equiv \mathrm{i}\left(\zeta\left.\frac{\partial}{\partial\zeta}\right|_{\bar\zeta} - \bar\zeta\left.\frac{\partial}{\partial\bar\zeta}\right|_\zeta\right), \tag{3.7}$$

the operator in (2.2) can be expressed in terms of the complex variables ζ and $\bar\zeta$:

$$\nabla^2_{\mathcal{T}_{R,r}} \equiv \frac{4|\zeta|^2}{(R - r\cos\theta)^2}\frac{\partial^2}{\partial\zeta\partial\bar\zeta}. \tag{3.8}$$

It remains to write $(R - r\cos\theta)^2$ in terms of ζ and $\bar\zeta$. From (3.4) and (3.6), we have

$$|\zeta| = \exp\left(-2\mathcal{A}\tan^{-1}\left(\sqrt{\frac{\alpha+1}{\alpha-1}}\tan\left(\frac{\theta}{2}\right)\right)\right). \tag{3.9}$$

from which it is possible to deduce that

$$\cos\theta = \alpha\left(\frac{\eta^2 + \frac{2}{\alpha}\eta + 1}{\eta^2 + 2\alpha\eta + 1}\right) \tag{3.10}$$

and hence that

$$F(\zeta,\bar\zeta) \equiv R - r\cos\theta = \frac{2r\eta/\mathcal{A}^2}{\eta^2 + 2\alpha\eta + 1}, \tag{3.11}$$

where $\eta \equiv \eta(\zeta,\bar\zeta)$ is defined as

$$\eta(\zeta,\bar\zeta) \equiv |\zeta|^{\mathrm{i}/\mathcal{A}}. \tag{3.12}$$

Thus, the Laplace–Beltrami operator in terms of the complex variables ζ and $\bar\zeta$ is

$$\nabla^2_{\mathcal{T}_{R,r}} \equiv \frac{4|\zeta|^2}{F^2(\zeta,\bar\zeta)}\frac{\partial^2}{\partial\zeta\partial\bar\zeta} \tag{3.13}$$

with function $F(\zeta,\bar\zeta)$ as defined in (3.11).

We point out that the function

$$h(\zeta,\bar\zeta) \equiv \frac{F(\zeta,\bar\zeta)}{|\zeta|} \tag{3.14}$$

appearing in (3.13) is the conformal factor, which gives

$$\mathrm{d}s^2 = h^2(\zeta,\bar\zeta)(\mathrm{d}q^2 + q^2\,\mathrm{d}\phi^2), \tag{3.15}$$

where $\mathrm{d}s$ denotes an element of length on the surface of the torus.

4. Construction of solution

Let us write Green's function $\psi(\zeta,\bar{\zeta})$, to be determined, as the sum of two real-valued functions

$$\psi(\zeta,\bar{\zeta}) = \psi_0(\zeta,\bar{\zeta}) + \psi_1(\zeta,\bar{\zeta}), \tag{4.1}$$

where $\psi_0(\zeta,\bar{\zeta})$ is the solution to

$$\nabla^2_{\mathcal{T}_{R,r}}\psi_0(\zeta,\bar{\zeta}) = \delta(\zeta-\zeta_0) \tag{4.2}$$

and $\psi_1(\zeta,\bar{\zeta})$ is the solution to

$$\nabla^2_{\mathcal{T}_{R,r}}\psi_1(\zeta,\bar{\zeta}) = -\frac{1}{4\pi^2 rR}. \tag{4.3}$$

Here ζ_0 corresponds to (θ_0,ϕ_0) under the stereographic projection (3.4). Note that the existence of $\psi_0(\zeta,\bar{\zeta})$ and $\psi_1(\zeta,\bar{\zeta})$ satisfying (4.2) and (4.3) does not contradict Gauss's divergence theorem if we allow them both to be multi-valued.

(a) Schottky–Klein prime function

We now introduce a special transcendental function known as the Schottky–Klein prime function associated with the annulus D_ζ. It can be defined through the infinite product

$$P(\zeta,\rho) = (1-\zeta)\prod_{j=1}^{\infty}(1-\rho^j\zeta)(1-\rho^j\zeta^{-1}). \tag{4.4}$$

This function has simple zeroes at $\zeta=\rho^n$, $n\in\mathbb{Z}$, and satisfies the functional relation

$$P(\rho\zeta,\rho) = -\zeta^{-1}P(\zeta,\rho). \tag{4.5}$$

An accessible overview of the Schottky–Klein prime function is given by Crowdy [27]. We also introduce another special function that is related to $P(\zeta,\rho)$ by

$$K(\zeta,\rho) = \frac{\zeta P'(\zeta,\rho)}{P(\zeta,\rho)} \tag{4.6}$$

where $P'(\zeta,\rho)$ denotes the derivative of $P(\zeta,\rho)$ with respect to the first argument. This function has simple poles at $\zeta=\rho^n$, $n\in\mathbb{Z}$ (i.e. at the simple zeroes of $P(\zeta,\rho)$), and satisfies the functional relation

$$K(\rho\zeta,\rho) = K(\zeta,\rho) - 1, \tag{4.7}$$

i.e. it is quasi-periodic as $\zeta\mapsto\rho\zeta$.

(b) The derivative $\partial\psi(\zeta,\bar{\zeta})/\partial\zeta$

We first construct the derivative $\partial\psi(\zeta,\bar{\zeta})/\partial\zeta$, as follows.

Let us first find $\partial\psi_0(\zeta,\bar{\zeta})/\partial\zeta$. We propose

$$\frac{\partial\psi_0(\zeta,\bar{\zeta})}{\partial\zeta} = \frac{K(\zeta/\zeta_0,\rho)}{4\pi\zeta} + \frac{\gamma}{2\zeta}, \tag{4.8}$$

for some $\gamma\in\mathbb{C}$. We can check that (4.8) satisfies (4.2) as follows. Let T denote the surface of the torus $\mathcal{T}_{R,r}$ and let $\mathrm{d}A$ be an area element on T. Let $\mathrm{d}\sigma$ be an area element in the ζ-plane. We must have

$$\iint_T \nabla^2_{\mathcal{T}_{R,r}} \psi_0(\theta,\phi)\,\mathrm{d}A = \iint_T \delta(\theta,\phi;\theta_0,\phi_0)\,\mathrm{d}A = 1. \tag{4.9}$$

By transforming variables $(\theta,\phi) \to (\zeta,\bar{\zeta})$, it can be shown that

$$\iint_T \nabla^2_{\mathcal{T}_{R,r}} \psi_0(\theta,\phi)\,\mathrm{d}A = 4\iint_{D_\zeta} \frac{\partial^2\psi_0(\zeta,\bar{\zeta})}{\partial\zeta\,\partial\bar{\zeta}}\,\mathrm{d}\sigma. \tag{4.10}$$

By Green's theorem

$$\iint_{D_\zeta} \frac{\partial^2\psi_0(\zeta,\bar{\zeta})}{\partial\zeta\,\partial\bar{\zeta}}\,\mathrm{d}\sigma = -\frac{\mathrm{i}}{2}\oint_{\partial D_\zeta} \frac{\partial\psi_0(\zeta,\bar{\zeta})}{\partial\zeta}\,\mathrm{d}\zeta \tag{4.11}$$

where ∂D_ζ denotes the boundary of D_ζ. Hence, we require

$$\oint_{\partial D_\zeta} \frac{\partial\psi_0(\zeta,\bar{\zeta})}{\partial\zeta}\,\mathrm{d}\zeta = \frac{\mathrm{i}}{2}. \tag{4.12}$$

Invoking the Argument Principle, and the fact that $K(\zeta/\zeta_0,\rho)$ has a single simple pole in D_ζ at $\zeta=\zeta_0$ with residue 1, it can be shown that

$$\oint_{\partial D_\zeta} \left(\frac{K(\zeta/\zeta_0,\rho)}{4\pi\zeta} + \frac{\gamma}{2\zeta}\right)\mathrm{d}\zeta = \frac{\mathrm{i}}{2}. \tag{4.13}$$

This completes our check of (4.8).

We now solve for $\partial\psi_1(\zeta,\bar{\zeta})/\partial\zeta$. It follows immediately from (4.3) that

$$\frac{\partial\psi_1(\zeta,\bar{\zeta})}{\partial\zeta} = -\frac{1}{16\pi^2 rR\zeta}\int^{\bar{\zeta}} \frac{F^2(\zeta,\overline{\zeta'})}{\overline{\zeta'}}\,\mathrm{d}\overline{\zeta'}. \tag{4.14}$$

It can be shown that

$$\begin{aligned}\xi(\zeta,\bar{\zeta}) &\equiv -\frac{1}{16\pi^2 rR}\int^{\bar{\zeta}} \frac{F^2(\zeta,\overline{\zeta'})}{\overline{\zeta'}}\,\mathrm{d}\overline{\zeta'}\\ &= -\frac{\mathrm{i}}{8\pi^2}\left[\log\left(\frac{\eta-c}{\eta-c^{-1}}\right) + \frac{1}{\alpha\mathcal{A}}\left(\frac{c}{\eta-c} + \frac{c^{-1}}{\eta-c^{-1}}\right)\right] + \varsigma_1(\zeta),\end{aligned} \tag{4.15}$$

where η is as in (3.12), $\varsigma_1(\zeta)$ is an arbitrary function of ζ and c is the constant given by

$$c = -\left(\alpha + \frac{1}{\mathcal{A}}\right). \tag{4.16}$$

c and $1/c$ are the roots of $\eta^2 + 2\alpha\eta + 1 = 0$. Note that because $R > r$, it is evident that $c \in \mathbb{R}$ and $c < -1$. We have thus found that

$$\frac{\partial\psi_1(\zeta,\bar{\zeta})}{\partial\zeta} = \frac{\xi(\zeta,\bar{\zeta})}{\zeta}. \tag{4.17}$$

By adding (4.8) and (4.17), we see that the derivative of Green's function $\psi(\zeta,\bar{\zeta})$ with respect to ζ is thus

$$\frac{\partial\psi(\zeta,\bar{\zeta})}{\partial\zeta} = \frac{K(\zeta/\zeta_0,\rho)}{4\pi\zeta} + \frac{\gamma}{2\zeta} + \frac{\xi(\zeta,\bar{\zeta})}{\zeta}. \tag{4.18}$$

(c) Green's function $\psi(\zeta,\bar{\zeta})$

We now turn our attention to finding an explicit expression for Green's function $\psi(\zeta,\bar{\zeta})$ using the result (4.18) of the previous section.

Integrating (4.8) with respect to ζ yields

$$\psi_0(\zeta,\bar{\zeta}) = \frac{1}{2\pi}\log\left|P\left(\frac{\zeta}{\zeta_0},\rho\right)\right| + \gamma\log|\zeta|. \tag{4.19}$$

Here, we have chosen the arbitrary function of $\bar{\zeta}$ of integration in order that $\psi_0(\zeta,\bar{\zeta})$ is indeed a purely real-valued function.

Next, integrating (4.17) with respect to ζ:

$$\psi_1(\zeta,\bar{\zeta}) = \int^{\zeta} \frac{\xi(\zeta',\bar{\zeta})}{\zeta'}\,\mathrm{d}\zeta'. \tag{4.20}$$

The integral in (4.20) can be computed using (4.15) as follows. We shall make use of the dilogarithm function defined by:

$$\mathrm{Li}_2(\zeta) = -\int_0^{\zeta} \frac{\log(1-u)}{u}\,\mathrm{d}u. \tag{4.21}$$

$\mathrm{Li}_2(\zeta)$ is an analytic function of ζ except for branch points at 1 and ∞. We choose a branch cut along the section of the real line $[1,\infty)$ in order to make it single-valued. A comprehensive discussion of the various properties of the dilogarithm function is given by Maximon [25]. Writing

$$\log\left(\frac{\eta-c}{\eta-c^{-1}}\right) = \log(1-c^{-1}\eta) - \log(1-c^{-1}\eta^{-1}) - \log\eta + \log(-c), \tag{4.22}$$

it is possible to show that

$$\begin{aligned}&\int^{\zeta} \log\left(\frac{\eta-c}{\eta-c^{-1}}\right)\frac{\mathrm{d}\zeta'}{\zeta'}\\ &\quad= 2\mathrm{i}\mathcal{A}\left(\mathrm{Li}_2(c^{-1}\eta) + \mathrm{Li}_2(c^{-1}\eta^{-1}) + \tfrac{1}{2}(\log\eta)^2 - \log(-c)\log\eta\right) + \varsigma_2(\bar{\zeta}).\end{aligned} \tag{4.23}$$

Also,

$$\int^{\zeta}\left(\frac{c}{\eta-c} + \frac{-1}{\eta-c^{-1}}\right)\frac{\mathrm{d}\zeta'}{\zeta'} = -2\mathrm{i}\mathcal{A}(\log(\eta-c) + \log(\eta-c^{-1}) - 2\log\eta) + \varsigma_3(\bar{\zeta}). \tag{4.24}$$

In (4.23) and (4.24), $\varsigma_2(\bar{\zeta})$ and $\varsigma_3(\bar{\zeta})$ are arbitrary functions of $\bar{\zeta}$. Now, note the following property of $\mathrm{Li}_2(\zeta)$, which can be deduced from its Taylor series expansion [25]:

$$\mathrm{Li}_2(\bar{\zeta}) = \overline{\mathrm{Li}_2(\zeta)}, \quad \text{for all } \zeta \text{ such that } |\zeta| < 1. \tag{4.25}$$

From (3.12), it may be deduced that $|\eta| = 1$ for all ζ. Recall also that $|c| > 1$ and so $|c^{-1}\eta| < 1$ for all ζ. Hence, using (4.25) and the fact that $c \in \mathbb{R}$, we have

$$\mathrm{Li}_2(c^{-1}\eta) + \mathrm{Li}_2(c^{-1}\eta^{-1}) = 2\mathrm{Re}\{\mathrm{Li}_2(c^{-1}\eta)\}. \tag{4.26}$$

Furthermore, again using the facts that $|\eta| = 1$ and $c \in \mathbb{R}$, we have

$$\log(\eta-c) + \log(\eta-c^{-1}) = 2\log|\eta-c| + \log\eta - \log(-c). \tag{4.27}$$

By combining the above, we arrive at

$$\psi_1(\zeta,\bar{\zeta}) = \lambda_1\mathrm{Re}\{\mathrm{Li}_2(c^{-1}\eta)\} + \lambda_2\log|\eta-c| + \lambda_3(\log|\zeta|)^2 + \lambda_4\log|\zeta| + \lambda_5 \tag{4.28}$$

where $\{\lambda_j | j = 1,\ldots,5\} \in \mathbb{R}$ are constants (i.e. independent of both ζ and $\bar{\zeta}$), with

$$\lambda_1 = \frac{\mathcal{A}}{2\pi^2}, \quad \lambda_2 = -\frac{1}{2\pi^2\alpha} \quad \text{and} \quad \lambda_3 = -\frac{1}{8\pi^2\mathcal{A}}. \tag{4.29}$$

λ_4 and λ_5 are yet to be specified owing to the arbitrary function of ζ in (4.15) and the arbitrary functions of $\bar{\zeta}$ in (4.23) and (4.24). We can determine λ_4 and λ_5 as follows. First recall that $\psi_1(\zeta,\bar{\zeta})$ must be a real-valued function. Using this fact, it follows from (4.28) that λ_4 and λ_5 must both be real.

Furthermore, the complete Green's function $\psi(\zeta,\bar{\zeta})$ must be single-valued on the torus and hence invariant with respect to the transformations $\zeta \mapsto \mathrm{e}^{2\pi\mathrm{i}}\zeta$ and $\zeta \mapsto \rho\zeta$. It is straightforward to check that for any choice of λ_4 and λ_5, the right-hand side of (4.28) is invariant with respect to $\zeta \mapsto e^{2\pi i}\zeta$. Now consider $\zeta \mapsto \rho\zeta$. It follows from (4.19) and (4.5) that

$$\psi_0(\rho\zeta, \rho\bar{\zeta}) = \psi_0(\zeta, \bar{\zeta}) - \frac{1}{2\pi}\log|\zeta| \tag{4.30}$$

provided we make the choice

$$\gamma = \frac{\log|\zeta_0|}{4\pi^2\mathcal{A}}. \tag{4.31}$$

Next, it can be deduced from (3.12) that $\zeta \mapsto \rho\zeta$ corresponds to $\eta \mapsto e^{-2\pi i}\eta$. Because $|\eta| = 1$ for all ζ, $\zeta \mapsto \rho\zeta$ corresponds to η, moving once in the clockwise direction around the unit η-circle. Note that because $|c| > 1$, then $|c^{-1}\eta| < 1$, and so as $\eta \mapsto e^{-2\pi i}\eta$, $c^{-1}\eta$ does not go through the branch cut along $[1, \infty)$ associated with the dilogarithm function so the first term of (4.28) does not change. It follows from (4.28) that

$$\psi_1(\rho\zeta, \rho\bar{\zeta}) = \psi_1(\zeta, \bar{\zeta}) + \lambda_3(2\log\rho\log|\zeta| + (\log\rho)^2) + \lambda_4\log\rho. \tag{4.32}$$

Hence, adding (4.30) and (4.32), it follows that we must have

$$\lambda_4 = -\frac{1}{4\pi}. \tag{4.33}$$

Finally, $\lambda_5 \in \mathbb{R}$ but may otherwise be chosen arbitrarily. We may wish to choose λ_5 so that the complete Green's function satisfies a reciprocity condition.

Green's function for the operator $\nabla^2_{\mathcal{T}_{R,r}}$ is thus found to be

$$\begin{aligned}\psi(\zeta, \bar{\zeta}) = \frac{1}{2\pi}\log\left|P\left(\frac{\zeta}{\zeta_0}, \rho\right)\right| &+ \lambda_1 \mathrm{Re}\{\mathrm{Li}_2(c^{-1}|\zeta|^{i/\mathcal{A}})\} \\ &+ \lambda_2\log||\zeta|^{i/\mathcal{A}} - c| + \lambda_3(\log|\zeta|)^2 + \left(\gamma - \left(\frac{1}{4\pi}\right)\right)\log|\zeta|\end{aligned} \tag{4.34}$$

with $\{\lambda_j | j = 1, \dots, 3\}$ as in (4.29), γ as in (4.31), c as in (4.16), $\mathcal{A}$ as in (3.12) and $\lambda_5 = 0$ (say).

We point out that it is straightforward to write

$$\log\left||\zeta|^{i/\mathcal{A}} - c\right|^2 = -\log h(\zeta, \bar{\zeta}) - \log|\zeta| + \text{const.} \tag{4.35}$$

It follows from well-known results of differential geometry that $-\nabla^2_{\mathcal{T}_{R,r}}\log h(\zeta, \bar{\zeta})$ equals the Gaussian curvature of the toroidal surface. This gives a geometrical interpretation to one of the terms appearing in (4.34).

Finally, we mention that on differentiation of (4.34) with respect to ζ, the function $\varsigma_1(\zeta)$ appearing in (4.15) may be identified as the constant

$$\varsigma_1(\zeta) = \frac{\gamma}{2} + \frac{i}{8\pi^2}\left(\log c - \frac{1}{\alpha\mathcal{A}}\right). \tag{4.36}$$

Figure 1 shows contour plots of $\psi(\zeta, \bar{\zeta})$ in D_ζ for the torus $\mathcal{T}_{4,1}$. We chose three distinct values of the singularity $\zeta = \zeta_0$, and took them to be real and positive (without loss of generality). The location of the singularity $\zeta = \zeta_0$ in each contour plot is apparent. For each value of ζ_0, we observe the existence of a critical point on the negative real axis, and two saddle points: one located on the positive real axis, and another located on the negative real axis close to the boundary. Inspection of the eigenvalues of the Hessian matrix of $\psi(\zeta, \bar{\zeta})$ confirms the existence of the two saddle points, and reveals that the critical point is in fact a maximum. By our choice of sign of $\psi(\zeta, \bar{\zeta})$, $\zeta = \zeta_0$ is a minimum.

Numerical experiments seem to suggest that under a change of the value of α, the number of saddle and critical points remains the same. However, changing α does have an effect on the qualitative appearance of the contours. More specifically, as $\alpha \to 1$, it appears that, for certain positions of the singularity ζ_0, there exist contours in the ζ-plane that encircle the inner boundary of D_ζ. These correspond to contours on the torus encircling its hole. As an example, illustrated in figure 2 are contours for the torus $\mathcal{T}_{1.6,1}$, with $\zeta_0 = 0.5$. However, because ρ is very small for α

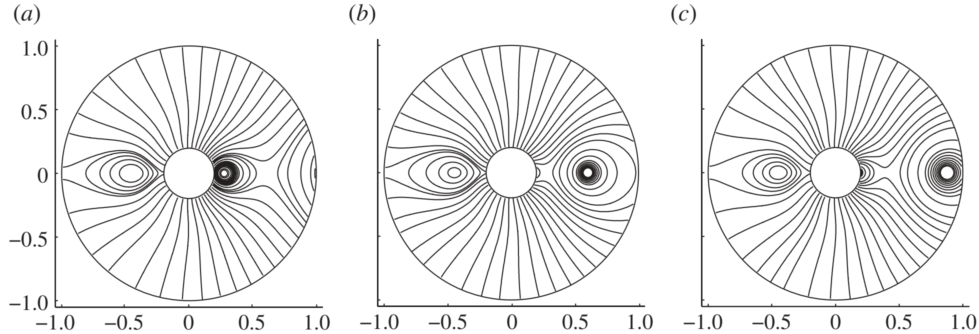

Figure 1. Contour plots of $\psi(\zeta, \bar{\zeta})$ in D_ζ for the torus $\mathcal{T}_{4,1}$ (for which $\rho = 0.19744$), for three distinct values of the singularity $\zeta = \zeta_0$. (*a*) $\zeta_0 = 0.275$, (*b*) $\zeta_0 = 0.6$, (*c*) $\zeta_0 = 0.875$.

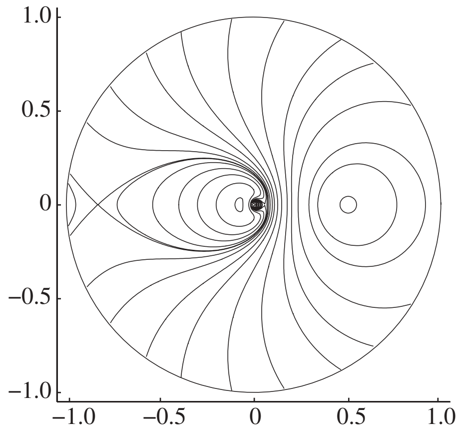

Figure 2. Contour plot of $\psi(\zeta, \bar{\zeta})$ in D_ζ for the torus $\mathcal{T}_{1.6,1}$ (for which $\rho = 0.00654$) with $\zeta_0 = 0.5$. One observes contours surrounding the inner boundary circle.

close to 1 (for this particular case, $\rho = 0.00654$), this behaviour is rather difficult to discern from the given plot. On the other hand, for $\alpha \gg 1$, changing α does not appear to affect the appearance of the contours. Figure 3 shows contours for the torus $\mathcal{T}_{8,1}$, with $\zeta_0 = 0.7$. One observes no qualitative difference in their appearance compared with those in figure 1. Similar behaviour is observed for larger α. Finally, we point out that our numerical experiments have been non-exhaustive, and a rigorous analysis of the behaviour of Green's function for the torus remains to be performed.

5. Application to vortex dynamics

In this section, we present an application of the foregoing theory to the field of vortex dynamics. We consider the flow of an infinitesimally thin layer of inviscid, incompressible fluid on $\mathcal{T}_{R,r}$. As will be shown, the Green's function we found in (4.34) is the streamfunction describing such a flow with a single point vortex surrounded by a uniform distribution of vorticity. The interested reader is referred to the monograph of Saffman [28] for an explanation of the key concepts used below.

The velocity field of the fluid $\mathbf{u}$ on $\mathcal{T}_{R,r}$ is purely tangential to the surface; that is, in terms of the coordinates (r, θ, ϕ), we have

$$\mathbf{u} = (0, u_\theta, u_\phi), \tag{5.1}$$

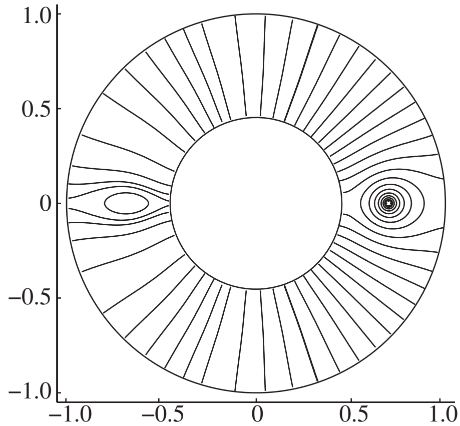

Figure 3. Contour plot of $\psi(\zeta,\bar{\zeta})$ in D_ζ for the torus $\mathcal{T}_{8,1}$ (for which $\rho = 0.453$) with $\zeta_0 = 0.7$. The contours are qualitatively the same as those in figure 1.

for some u_θ, u_ϕ. Owing to the incompressibility condition $\nabla_{\mathcal{T}_{R,r}} \cdot \mathbf{u} = 0$, we may introduce the streamfunction $\psi(\theta,\phi)$ such that

$$\mathbf{u} = \nabla_{\mathcal{T}_{R,r}} \psi(\theta,\phi) \times (1,0,0) = \left(0, \frac{1}{R - r\cos\theta}\frac{\partial\psi}{\partial\phi}, -\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right). \tag{5.2}$$

It follows from (3.7) and (5.2) that, in terms of ζ, the velocity field can be expressed in the form

$$u_\phi - \mathrm{i}u_\theta = \frac{2\zeta}{F(\zeta,\bar{\zeta})}\frac{\partial\psi}{\partial\zeta}. \tag{5.3}$$

Furthermore, it can be shown that the 'image' flow in the ζ-plane is given by

$$U - \mathrm{i}V = -\frac{2\mathrm{i}|\zeta|^2}{F^2(\zeta,\bar{\zeta})}\frac{\partial\psi}{\partial\zeta}, \tag{5.4}$$

where U and V denote, respectively, the components of the velocity in the real and imaginary directions.

Now introduce the scalar vorticity field $\omega(\theta,\phi)$ defined by

$$\omega(\theta,\phi) = (\nabla_{\mathcal{T}_{R,r}} \times \mathbf{u}) \cdot (1,0,0). \tag{5.5}$$

Taking the curl of (5.2) and equating with (5.5) yields

$$\nabla^2_{\mathcal{T}_{R,r}} \psi(\theta,\phi) = -\omega(\theta,\phi). \tag{5.6}$$

Because $\mathcal{T}_{R,r}$ is a closed compact surface, it follows from Gauss's Divergence Theorem that we have an intrinsic topological constraint to enforce on $\mathbf{u}$; that is,

$$\iint_T \omega(\theta,\phi)\,\mathrm{d}A = 0. \tag{5.7}$$

Consider now a point vortex on $\mathcal{T}_{R,r}$. This provides a δ-function distribution of vorticity. As a consequence of (5.7), a single point vortex cannot exist on $\mathcal{T}_{R,r}$ unless an additional source of vorticity is present. One way to resolve this is as follows. Suppose the point vortex has circulation -1. By endowing the torus with a background sea of uniform vorticity $\omega_0 = 1/4\pi^2 rR$, the circulation associated with the point vortex is nullified; thus, the net circulation on $\mathcal{T}_{R,r}$ will automatically be zero and the condition (5.7) satisfied. The streamfunction $\psi(\theta,\phi)$ for such a system will now be deduced.

Suppose the point vortex is at (θ_0,ϕ_0) on the surface of $\mathcal{T}_{R,r}$. Then $\psi(\theta,\phi)$ satisfies the Laplace–Beltrami equation (2.3), in which physically, the Dirac delta function $\delta(\theta,\phi;\theta_0,\phi_0)$ corresponds to the point vortex at (θ_0, ϕ_0) while the additive constant term corresponds to the sea of uniform vorticity ω_0 covering the surface of the torus. We thus identify $\psi(\theta, \phi)$ as Green's function of $\nabla^2_{\mathcal{T}_{R,r}}$ found in (4.34).

Consider now the velocity field associated with such a vorticity distribution. It follows from (5.3) and (4.18) that this is given by

$$u_\phi - \mathrm{i}u_\theta = \frac{2}{F(\zeta,\bar{\zeta})}\left(\frac{1}{4\pi}K(\zeta/\zeta_0,\rho) + \frac{\gamma}{2} + \xi(\zeta,\bar{\zeta})\right), \tag{5.8}$$

where ζ_0 is the image of the point vortex in the ζ-plane, and functions $K(\zeta,\rho)$ and $\xi(\zeta,\bar{\zeta})$ are as in (4.6) and (4.15), respectively, recalling also that $\varsigma_1(\zeta)$ is given by (4.36). Both $K(\zeta,\rho)$ and $\xi(\zeta,\bar{\zeta})$ are quasi-periodic as $\zeta \mapsto \rho\zeta$. We remark that there cannot exist doubly periodic functions in a period rectangle (and hence the equivalent concentric annulus) corresponding to the contributions to the velocity field of either the point vortex or the uniform sea of vorticity. We would thus expect a total velocity field to be the sum of two quasi-periodic functions: one corresponding to the point vortex, and one corresponding to the sea of uniform vorticity.

On physical grounds, however, the velocity field (5.8) must be single-valued on $\mathcal{T}_{R,r}$ and hence invariant as $\zeta \mapsto \rho\zeta$ and $\zeta \mapsto \mathrm{e}^{2\pi\mathrm{i}}\zeta$, or equivalently, as $\eta \mapsto \mathrm{e}^{-2\pi\mathrm{i}}\eta$ and $\eta \mapsto \eta$, respectively. We now demonstrate this explicitly. First, note that the function $F(\zeta,\bar{\zeta})$ is invariant with respect to both transformations. It is straightforward to check that (5.8) is invariant with respect to $\zeta \mapsto \mathrm{e}^{2\pi\mathrm{i}}\zeta$. Next, consider this expression subject to $\zeta \mapsto \rho\zeta$. Recall that $c \in \mathbb{R}$ and $|c| > 1$. Then, from (4.15), it may be deduced that

$$\xi(\rho\zeta, \bar{\rho\zeta}) = \xi(\zeta,\bar{\zeta}) + \frac{1}{4\pi}. \tag{5.9}$$

It thus follows from (4.7) and (5.9) that (5.8) is also invariant with respect to $\zeta \mapsto \rho\zeta$, as required.

Finally, in the ζ-plane, we have from (5.4) that

$$U - \mathrm{i}V = -\frac{2\mathrm{i}|\zeta|^2}{F^2(\zeta,\bar{\zeta})}\left(\frac{K(\zeta/\zeta_0,\rho)}{4\pi\zeta} + \frac{\gamma}{2\zeta} + \frac{\xi(\zeta,\bar{\zeta})}{\zeta}\right). \tag{5.10}$$

We remark that in (5.10), the simple pole of $K(\zeta/\zeta_0,\rho)$ at $\zeta = \zeta_0$ gives rise to a singularity that is reminiscent of a planar point vortex. The pre-multiplying factor incorporates the curvature of the toroidal surface, and thus generalizes this singularity so that it pertains to a point vortex on a toroidal surface. Because the curvature of the toroidal surface is non-constant, it is not obvious whether the point vortex itself is advected by the velocity field (5.10). This issue is raised and discussed for a general closed surface in Boatto & Koiller [10]. However, clarification of this issue for the particular case of the torus remains a matter for future investigation.

6. Discussion

In this paper, we have derived, using techniques of complex analysis, an expression for Green's function of the Laplace–Beltrami operator on a toroidal surface. By using a stereographic projection to a planar concentric annulus, and performing the analysis in this projected plane, our Green's function admits a particularly concise form in terms of two special functions—the Schottky–Klein prime function and the dilogarithm function—yet still fully appreciates the topology of the torus.

The Schottky–Klein prime function is a natural candidate when considering problems in multiply connected circular domains. Having opted to construct Green's function in a complex plane, it was therefore natural to stereographically project the torus onto a concentric annulus and readily exploit the special properties of the Schottky–Klein prime function and its associated function theory in this circular domain. Furthermore, the related theory of Schottky groups naturally extends to provide uniformizations of more general higher genus surfaces. It may be possible to construct Green's functions for other surfaces in these terms by generalizing the analysis presented here.

It may be interesting to consider our formulation for Green's function in the limiting case as the major radius R becomes infinite with the minor radius r remaining fixed. Then, locally, the toroidal surface appears to be that of an infinitely long cylinder. In this limit, $\rho \to 1$ and the inner boundary of the annulus D_ζ approaches the outer. It may be more helpful to consider the analysis in this case in the rectangle in the conformally equivalent Z-plane, re-scaling by a factor of R so that this rectangle tends to an infinite strip of finite thickness. We may then use the Weierstrass elliptic functions that are related to $P(\zeta, \rho)$ and $K(\zeta, \rho)$ in a straightforward manner.

Green's function we have constructed should have a range of physical interpretations when considered as a solution to other problems of physical interest where the Laplace–Beltrami equation on a toroidal surface governs the system. For example, we have shown that the streamfunction for a point vortex on a toroidal surface is precisely Green's function constructed. Our Green's function should prove to be a convenient starting point for investigating these problems.

Keeping with the application to vortex dynamics, it would be interesting to emulate the various systems on a sphere considered by various authors [2,3,7,8,15–17] but on a toroidal surface. It would be of interest to analyse the possible similarities and differences of the exhibiting behaviours of the vortical systems between compact surfaces of differing genus. Such lines of enquiry are matters for future investigation.

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References

1. Kimura Y, Okamoto H. 1987 Vortex motion on a sphere. *J. Phys. Soc. Jpn* **56**, 4203–4206. (doi:10.1143/JPSJ.56.4203)
2. Kimura Y. 1999 Vortex motion on surfaces with constant curvature. *Proc. R. Soc. Lond. A* **455**, 245–259. (doi:10.1098/rspa.1999.0311)
3. Crowdy DG, Cloke M. 2003 Analytical solutions for distributed multipolar vortex equilibria on a sphere. *Phys. Fluids* **15**, 22–34. (doi:10.1063/1.1521727)
4. Bogomolov VA. 1977 Dynamics of vorticity at a sphere. *Fluid Dyn.* **6**, 863–870. (doi:10.1007/BF01090320)
5. Boatto S, Simó C. 2008 Thomson's heptagon: a case of bifurcation at infinity. *Physica D* **237**, 2051–2055. (doi:10.1016/j.physd.2008.02.030)
6. Kidambi R, Newton PK. 1998 Motion of three point vortices on a sphere. *Physica D* **116**, 143–175. (doi:10.1016/S0167-2789(97)00236-4)
7. Kidambi R, Newton PK. 2000 Point vortex motion on a sphere with solid boundaries. *Phys. Fluids* **12**, 581–588. (doi:10.1063/1.870263)
8. Kim S-C. 2010 Latitudinal point vortex rings on the spheroid. *Proc. R. Soc. A* **466**, 1749–1768. (doi:10.1098/rspa.2009.0597)
9. Hally D. 1980 Stability of streets of vortices on surfaces of revolution with a reflection symmetry. *J. Math. Phys.* **21**, 211–217. (doi:10.1063/1.524322)
10. Boatto S, Koiller J. 2008 Vortices on closed surfaces. Preprint, pp. 1–16. See http://arxiv.org/pdf/0802.4313.pdf.
11. Turner AM, Vitelli V, Nelson DR. 2010 Vortices on curved surfaces. *Rev. Mod. Phys.* **82**, 1301–1348. (doi:10.1103/RevModPhys.82.1301)
12. Corrada-Emmanuel A. 1994 Exact solution for superfluid film vortices on a torus. *Phys. Rev. Lett.* **72**, 681–684. (doi:10.1103/PhysRevLett.72.681)
13. Hardin DP, Saff EB. 2004 Discretizing manifolds via minimum energy points. *Notices of the AMS* **51**, 1186–1194.
14. Newton PK, Sakajo T. 2010 Point vortex equilibria and optimal packings of circles on a sphere. *Proc. R. Soc. A* **467**, 1468–1490. (doi:10.1098/rspa.2010.0368)
15. Crowdy DG. 2004 Stuart vortices on a sphere. *J. Fluid Mech.* **498**, 381–402. (doi:10.1017/S0022112003007043)
16. Crowdy DG. 2006 Point vortex motion on the surface of a sphere with impenetrable boundaries. *Phys. Fluids* **18**, 036602. (doi:10.1063/1.2183627)

17. Surana A, Crowdy DG. 2008 Vortex dynamics in complex domains on a spherical surface. *J. Comp. Phys.* **227**, 6058–6070. (doi:10.1016/j.jcp.2008.02.027)
18. Hasimoto H. 2008 Periodic fundamental solution of a two-dimensional Poisson equation. *J. Phys. Soc. Jpn* **77**, 104601. (doi:10.1143/JPSJ.77.104601)
19. Lin C-S, Wang C-L. 2010 Elliptic functions, Green functions and the mean field equations on tori. *Ann. Math.* **172**, 911–954. (doi:10.4007/annals.2010.172.911)
20. Boyland P, Stremler MA, Aref H. 2003 Topological fluid mechanics of point vortex motions. *Physica D* **175**, 69–95. (doi:10.1016/S0167-2789(02)00692-9)
21. Montaldi J, Soulière A, Tokieda T. 2003 Vortex dynamics on a cylinder. *SIAM J. Appl. Dyn. Syst.* **2**, 417–430. (doi:10.1137/S1111111102415569)
22. Soulière A, Tokieda T. 2002 Periodic motions of vortices on surfaces with symmetry. *J. Fluid Mech.* **460**, 83–92.
23. Stremler MA, Aref H. 1999 Motion of three point vortices in a periodic parallelogram. *J. Fluid Mech.* **392**, 101–128. (doi:10.1017/S002211209900542X)
24. Crowdy DG. 2010 On rectangular vortex lattices. *Appl. Math. Lett.* **23**, 34–38. (doi:10.1016/j.aml.2009.07.022)
25. Maximon LC. 2003 The dilogarithm function for complex argument. *Proc. R. Soc. Lond. A* **459**, 2807–2819. (doi:10.1098/rspa.2003.1156)
26. Akhiezer NI. 1990 *Elements of the theory of elliptic functions*. Providence, RI: American Mathematical Society.
27. Crowdy DG. 2008 Geometric function theory: a modern view of a classical subject. *Nonlinearity* **21**, 205–219. (doi:10.1088/0951-7715/21/10/T04)
28. Saffman PG. 1992 *Vortex dynamics*. Cambridge, UK: Cambridge University Press.